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The dual-projection approach for two-mode networks

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ABSTRACT

There have been two distinct approaches to two-mode data. The first approach is to project the data to one-mode and then analyze the projected network using standard single-mode techniques, also called the conversion method. The second approach has been to extend methods and concepts to the two-mode case and analyze the network directly with the two modes considered jointly. The direct approach in recent years has been the preferred method since it is assumed that the conversion method loses important structural information. Here we argue that this is not the case, provided both projections are used together in any analysis. We illustrate how this approach works using core/periphery, structural equivalence and centrality as examples.

1. Introduction

Broadly speaking, two basic approaches have been used to analyze two-mode data. Borgatti and Halgin (2011) refer to these as the "conversion" approach and the "direct" approach. In the conversion approach, one or both modes of the two-mode dataset are converted into two one-mode projections which are then analyzed separately. In the direct approach, the two-mode dataset is analyzed directly, with the two modes analyzed jointly. Conventional wisdom today seems to say that the direct approach is better. There are multiple reasons for this, but the key reason is probably that the conversion approach is assumed to lose important structural features of the data. In this paper, we examine this assumption, outlining the conditions under which it is true, and consider ways in which conversion-based analyses provide different insights relative to direct approaches. Specifically this paper has three goals. First, we show that data is not necessarily lost when we make projections, provided both projections are used in any subsequent analysis. Second, we identify existing methods that use both projections and so do not suffer from data loss. Third, we demonstrate that there is potential to develop new approaches and methods which utilize both projections and so increase the arsenal of techniques at our disposal for analyzing two-mode data.

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2. Terminology

We assume a two-mode matrix A with dimensions $n \times m$. The canonical example is the women-by-events matrix A collected by Davis et al. (1941) in which $a_{ij} = 1$ if woman i attended event j and $a_{ij} = 0$ otherwise. Based on this, we shall often refer to the row mode as persons and the column mode as events, but in actuality the row and column entities can be anything. We shall also usually assume that A is binary, but again this is for convenience only.

Given A, we can define two one-mode projections corresponding to the two modes of A, which we refer to as the row projection and the column projection. In the case of the Davis et al. data the row projection consists of a woman-by-woman matrix while the column projection consists of an event-by-event matrix. Canonically, these are defined by AA^T and A^TA respectively, although these can be regarded as just particular instances of a larger class of projections constituting similarities or dissimilarities of As row or column profiles. When defined as AA^T and A^TA, the row and column projections can be seen as overlap or "intersection" matrices (Breiger, 1974), such that, in the Davis et al. data, the entries of the row projection give the number of events that each pair of women attended in common, and the column projection gives the number of women that attended each pair of events.

3. Conversion versus direct analysis

The most obvious application of the conversion approach is when only one mode is of theoretical interest and the other is just used as an indicator of relationships among the entities of the primary mode. For example, Davis et al. (1941) were interested in the social structure among the women. They viewed co-attendance at events as evidence of an underlying relationship (dictated in part by social class).

Breiger (1974) noted that in principle we should always be interested in both the row and column projections, arguing that it is a fundamental tenet of social theory that a person's identity is defined by the intersection of the groups they belong to, while a group is defined by the intersection of persons who belong to it. He refers to this tenet as "dualism" and traces it to Simmel (1922) and Cooley (1902). He also notes on a more technical level that the row and column projections contain information about each other so that even if one's interest is only in one of the modes, the other mode's projection remains relevant.

The upshot of this has been that, in recent times, researchers tend to feel that we must analyze both modes together. But whereas Breiger meant that we should analyze both the row and column projections, the current view is that we must analyze the two-mode dataset directly in such a way that both modes are taken account of in an integrated way, without constructing the one-mode projections. The obvious approach (e.g., Borgatti and Everett, 1997) is to view the two-mode incidence matrix A as one quadrant of an $(m+n)$ by $(m+n)$ matrix which constitutes the adjacency matrix of a bipartite graph in which nodes from both modes are present, and all ties occur only between modes and not within. This new dataset is then analyzed using the usual network analysis techniques, with minor adjustments to deal with the special structure of bipartite graphs (Borgatti and Everett, 1997). For example, we can measure betweenness centrality for each node. As a result, the centrality of a woman in the Davis et al. dataset is very much a function of paths involving both other women and events, thus constituting an integrated handling of the two modes.

The idea of having the measurements of one mode depend so directly on the nodes of the other mode has definite appeal. But the real driver behind the preference for this direct approach is the belief that the conversion approach loses crucial structural features of the original two-mode dataset. This is of course true if we only analyze one projection, but we shall contend that using both one-mode projections together – which we refer to as the dual-projection approach – is a completely different matter that does not necessarily entail the loss of information.

However, we might ask what the benefit of a dual-projection approach is given that if we have both projections available, it is probably because we constructed from the original two-mode matrix, in which case we could simply have analyzed that directly. One very important reason is that there are only a limited number of social network analysis techniques that can be used on a rectangular two-mode data matrix. In addition, the bipartite representation results in a square matrix but does not keep the modes separate, which can cause a variety of problems, including theoretical ones (as in the case where one mode consists of active agents, like persons, while the other consists of passive things, like types of databases). The projections, on the other hand, are symmetric matrices – typically possessing many convenient mathematical properties – and each involves just one of the modes. This opens up the possibility of many more techniques than are available for direct analysis of two-mode data.[1]

4. Projection and data loss

The standard projections are formed by taking a two-mode binary matrix A and constructing AA^T and A^TA. There is no question that working with just AA^T or A^TA alone results in a loss of information. The question we ask is whether working with both together retains the information in A. Clearly, certain aspects of A are recoverable from the two projections used together. For example, suppose we had AA^T and A^TA and wanted to know the degree centrality of each node in the bipartite graph representation of A. Could we do it? Obviously, yes. The diagonal entries of the two projections give precisely that information. Similarly, Bonacich (1991) showed that we can compute the eigenvector centrality of each node in the bipartite graph from the two projections (we discuss this further in Section 8 where we give a proof of a slightly more general result). In these two cases we do not need the original data matrix in order to measure these structural variables.

The larger question is whether it is always the case that we can recover a given network property of the bipartite graph from the dual-projections. Or, put more fundamentally, given AA^T and A^TA, can we always reconstruct A? Of course, it is a given that we can only hope to reconstruct A up to an isomorphism of A. For example, if two persons attended exactly the same events it would be impossible to distinguish between them. More generally, any automorphically equivalent actors would be indistinguishable. This should not trouble us, however, since in that case they would have the same structural properties and no analysis would ever distinguish them. Hence, no conclusions would be altered by reconstructing the "wrong" A.

One way to disconfirm the conjecture is to find a case of two non-isomorphic two-mode matrices A and B that have the same projections. If they have the same projections, this would imply that differences in A and B were being lost in the conversion process, and given only the projections we could not determine whether they came from A or B. As it turns out, these are fairly easy to find because, given a symmetric matrix X with e non-zero eigenvalues, there are necessarily $2^e - 1$ other matrices with the same projections. To construct an example, choose an arbitrary diagonal matrix D with positive values down the diagonal. Take any orthogonal matrix P of the same size as D. Now construct $A = PDP^T$ and $B = PD^*P^T$, where D^* is D with any number of diagonal values made negative. Matrices A and B will be non-isomorphic, yet possess the same projections $AA^T = A^TA = BB^T = B^TB$.

Thus, it would appear that we cannot reconstruct A from AA^T and A^TA. Note, however, that in establishing this we have not constrained our starting matrices to be binary. It could still be that no two binary matrices could have identical projections and yet be non-isomorphic. Alas, this is not true either. Kirkland (private communication) has constructed one example in which $AA^T = A^TA$ and A is not isomorphic to A^T. However, the example is very special: a 19×19 matrix constructed from a Hadamard matrix. Hadamard matrices have a very regular structure (quite unlike the matrices we encounter in empirical research) and are extremely rare. In the same communication Kirkland has shown that the proportion of binary matrices that can have the same projections tends to zero as either matrix dimension tends to infinity. In short, while there are occasions when we cannot uniquely reconstruct A from its projections, it appears these cases involve very unusual matrices that are unlikely to occur in empirical settings and become vanishingly rare as matrix size increases. We conjecture that in the vast majority of empirical cases, analyzing the projections of a two-mode matrix does not entail any loss of information. This means that, in those cases, the two projections are a faithful representation of the data and are really just a different representation of the same information.

5. Singular value decomposition

One method that we can often employ to help us recover A from its projections is singular value decomposition, or SVD. This is a

[1] The authors are grateful to John P. Boyd, Nick Higham and Stephen Kirkland discussions on the issues in this section.

standard technique, and more details can be found in standard linear algebra books such as Numerical Linear Algebra by Trefethen and Bau (1997). We give a brief overview here.

For any matrix A, the separable form of the singular value decomposition states that

$$A = \lambda_1 \mathbf{u}_1 \mathbf{v}_1^{\mathrm{T}} + \lambda_2 \mathbf{u}_2 \mathbf{v}_2^{\mathrm{T}} + \cdots + \lambda_p \mathbf{u}_p \mathbf{v}_p^{\mathrm{T}} \quad (1)$$

where λ_i are the positive square roots of the non-zero eigenvalues of AA^{T} (or $A^{\mathrm{T}}A$), called singular values, the **u**s are eigenvectors of AA^{T}, and the **v**s are the eigenvectors of $A^{\mathrm{T}}A$, called singular vectors.

At first sight it would appear that, given AA^{T} and $A^{\mathrm{T}}A$, we could easily reconstruct A by finding the eigenvectors and eigenvalues of AA^{T} and $A^{\mathrm{T}}A$, which give us the λs, **u**s and **v**s. This is in fact true in some circumstances, although we must take care to deal with the fact that eigenvectors are not necessarily unique. First, the multiple of any eigenvector is also an eigenvector. We can deal with this in part by constraining our eigenvectors to, say, unit length. This still leaves a choice of sign, but in principle we can use trial and error to select the right signs. In this case we will have 2^{p-1} matrices to consider since half of them will have the same entries but be of opposite sign. Hence to reconstruct A from AA^{T} and $A^{\mathrm{T}}A$, we first use either of these to find the eigenvalues (the squares of the λs), and check that there are no repeated values. If not, we then find the set of unit-length eigenvectors of AA^{T}, giving us the **u**s, and find the unit-length eigenvectors of $A^{\mathrm{T}}A$, which give us the **v**s. Then we use the right-hand side of Eq. (1) to construct a matrix B (say). If B is a binary matrix then with very high probability it will be either the data matrix A or a matrix that is isomorphic to A. That is, it is just a permutation of the rows and columns of A. If it is not binary then we change the sign of all the entries in $\mathbf{u}_1$ and repeat the process. If this again is not binary then we change the sign of just $\mathbf{u}_2$ and again repeat. We continue this way and if changing the sign of one of the **u**s does not work, we try changing the sign of two **u**s. There will be up to 2^{p-1} possible combinations, and one of these will eventually yield a matrix that is isomorphic to A. If there are repeated eigenvalues then there will be sets of distinct eigenvectors (i.e., not multiples of each other) that correspond to the same eigenvalue. If we have two eigenvectors corresponding to the same eigenvalue, say, **r** and **t** (where **r** is not a multiple of **t**), then $\mathbf{r}+\mathbf{t}$ is also an eigenvector. This problem is not surmountable, but in practice it is fairly unusual to find real data matrices with repeated eigenvalues.[2]

Furthermore, a look at how we construct SVDs suggests that the amount of data loss is not as great as one might think. A common way of forming the right-hand side of Eq. (1) is to obtain the eigenvectors of either AA^{T} and $A^{\mathrm{T}}A$ and then combine the information with A itself to obtain the correct vectors. We note that Eq. (1) can be written in the following matrix form:

$$A = USV^{\mathrm{T}} \quad (2)$$

where A is an $m \times n$ matrix, U is $m \times p$, S is a diagonal matrix of size $p \times p$, V^{T} is $p \times n$, and the diagonal entries of S are the non-zero singular values. This is often called a reduced SVD. We can construct, say, V, from the eigenvectors associated with $A^{\mathrm{T}}A$, provided we select an orthonormal set and form S from the singular values. It is now easy to see that $U = AVS^{-1}$, and this makes it clear how A is used explicitly in the construction of U. The major computational work is done in forming V; inverting S is simple, as it is a diagonal matrix. In short, A is only minimally needed, as it is used in just one matrix multiplication. In addition, it should be noted that there may well be other methods for reconstructing A that do not require additional information from A. In fact, this is an active area of research in combinatorial matrices.

In summary, the last two sections have shown that we can reconstruct A from its projections, at least in those cases in which the singular values are unique. In such cases, it is clear that structural information about the original data is contained within the projections, and we are well justified in analyzing the data by forming the projections and combining the results from each. This does not imply that any particular set of techniques applied to the projections will necessarily match the results of a direct analysis of the two-mode data any more than the results of any two direct analyses will agree with each other. The point we are making is that techniques that rely on the one-mode projections almost always count on the same information that the direct methods do (and when they do not, we know it). The importance of this is that while some techniques lend themselves to the bipartite representation, others lend themselves to the one-mode projections. Knowing that in most cases the information in the two data representations is the same allows us to utilize the full repertoire of potential techniques. We should emphasize that we have considered the reconstruction of A from its two projections in order to gain an insight into data loss. We do not propose this as a method since we assume that A is available as our basic data and will be used to construct the projections.

In the next sections, we consider a set of projection-based methods, some of which can be matched to comparable two-mode methods, and some that provide a different perspective. We emphasize that, in order to reduce or eliminate any structural data loss, both projections must be used and the results combined. We shall call such an approach the dual-projection approach. The following sections give a few of the potential examples of how this works in practice.

6. Dual-projection approach to core/periphery models

Borgatti and Everett (1999) give two formal definitions of core/periphery structures, a discrete model and a continuous model. The discrete model is in the same vein as the optimization approach to blockmodels discussed by Batagelj et al. (1992a,b). The continuous model is based on a "coreness" score analogous to standard centrality measures. In fact, Borgatti and Everett show that eigenvector centrality is actually a core/periphery measure (which in itself can be thought of as a special type of centrality measure).

While the discrete model works well for binary-valued networks, it is not well suited to valued data. In contrast, the continuous model is extremely well suited to valued symmetric data such as that derived from the one-mode projections AA^{T} and $A^{\mathrm{T}}A$. We can therefore apply the standard continuous model implemented in UCINET (Borgatti et al., 2002) to each projection separately. This suggests the following approach. First, run the standard continuous model implemented in UCINET on each projection separately. Second, in each analysis, accept the recommended partition produced by UCINET to split core from periphery in both AA^{T} and $A^{\mathrm{T}}A$. This yields a core/periphery partition for each mode. These can then be combined to yield a two-mode blockmodel of the original two-mode data matrix.

Of course, the eigenvectors of AA^{T} and $A^{\mathrm{T}}A$ are simply the singular vectors from a singular value decomposition of A, which have already been proposed – in another context – by Boyd et al. (2010) as a method of identifying cores and peripheries. Taking note of the difficulties in applying the continuous core/periphery model to directed graphs, Boyd et al. suggest a singular value decomposition of the non-symmetric adjacency matrix, yielding separate in-coreness and out-coreness scores, which can then be partitioned into in-cores, in-peripheries, out-cores and out-peripheries. Thus,

[2] It is worth noting that any problem matrices based on Hadamard matrices, such as the example constructed by Kirkland, will have multiple repeated eigenvalues.

Table 1
Core–periphery partitions of both projections of the Davis data.

		1 E	2 L	3 T	4 B	14 N	7 E	9 R	13 S	6 F	8 P	10 V	12 K	11 M	5 C	15 H	16 D	17 O	18 F
1	EVELYN	8	6	7	6	2	3	3	2	4	3	2	2	2	3	1	2	1	1
2	LAURA	6	7	6	6	2	4	3	2	4	2	2	1	1	3	2	1		
3	THERESA	7	6	8	6	3	4	4	3	4	3	3	2	2	4	2	2	1	1
4	BRENDA	6	6	6	7	2	4	3	2	4	2	2	1	1	4	2	1		
14	NORA	2	2	3	2	8	2	2	6	1	2	3	5	3	1	4	1	2	2
7	ELEANOR	3	4	4	4	2	4	3	2	3	2	2	1	1	2	2	1		
9	RUTH	3	3	4	3	2	3	4	3	2	2	3	2	2	2	2	2	1	1
13	SYLVIA	2	2	3	2	6	2	3	7	1	2	4	6	4	1	4	2	1	1
6	FRANCES	4	4	4	4	1	3	2	1	4	2	1	1	1	2	1	1		
8	PEARL	3	2	3	2	2	2	2	2	2	3	2	2	2		1	2	1	1
10	VERNE	2	2	3	2	3	2	3	4	1	2	4	3	3	1	3	2	1	1
12	KATHERINE	2	1	2	1	5	1	2	6	1	2	3	6	4		3	2	1	1
11	MYRNA	2	1	2	1	3	1	2	4	1	2	3	4	4		3	2	1	1
5	CHARLOTTE	3	3	4	4	1	2	2	1	2		1			4	1			
15	HELEN	1	2	2	2	4	2	2	4	1	1	3	3	3	1	5	1	1	1
16	DOROTHY	2	1	2	1	1	1	2	2	1	2	2	2	2		1	2	1	1
17	OLIVIA	1		1		2		1	1		1	1	1	1		1	1	2	2
18	FLORA	1		1		2		1	1		1	1	1	1		1	1	2	2

		8 E8	9 E9	6 E6	7 E7	5 E5	3 E3	4 E4	1 E1	2 E2	10 E1	11 E1	12 E1	13 E1	14 E1
8	E8	14	9	7	8	7	5	3	3	3	4	1	5	2	2
9	E9	9	12	4	5	3	2	2	1	2	4	3	5	3	3
6	E6	7	4	8	5	6	5	3	3	3	1	1	1	1	1
7	E7	8	5	5	10	6	4	3	2	2	3	2	4	2	2
5	E5	7	3	6	6	8	6	4	3	3					
3	E3	5	2	5	4	6	6	4	3	3					
4	E4	3	2	3	3	4	4	4	2	2					
1	E1	3	1	3	2	3	3	2	3	2					
2	E2	3	2	3	2	3	3	2	2	3					
10	E10	4	4	1	3						5	2	5	3	3
11	E11	1	3	1	2						2	4	2	1	1
12	E12	5	5	1	4						5	2	6	3	3
13	E13	2	3	1	2						3	1	3	3	3
14	E14	2	3	1	2						3	1	3	3	3

Table 2
Core–periphery partition using projection splits on original Davis two-mode dataset.

		8 E	9 E	6 E	7 E	5 E	3 E	4 E	1 E	2 E	10 E	11 E	12 E	13 E	14 E
1	EVELYN	1	1	1		1	1	1	1	1					
2	LAURA	1		1	1	1	1		1	1					
3	THERESA	1	1	1	1	1	1	1		1					
4	BRENDA	1		1	1	1	1	1	1						
14	NORA		1	1	1						1	1	1	1	1
7	ELEANOR	1		1	1	1									
9	RUTH	1	1		1	1									
13	SYLVIA	1	1		1						1		1	1	1
6	FRANCES	1		1		1	1								
8	PEARL	1	1	1											
10	VERNE	1	1		1								1		
12	KATHERINE	1	1								1		1	1	1
11	MYRNA	1	1								1		1		
5	CHARLOTTE				1	1	1	1							
15	HELEN	1			1						1	1	1		
16	DOROTHY	1	1												
17	OLIVIA		1									1			
18	FLORA		1									1			

the conversion approach to the two-mode core/periphery problem is identical to the Boyd et al. approach to the directed one-mode case, yielding a single coherent approach for undirected one-mode graphs, directed one-mode graphs and two-mode affiliations.

We illustrate the method on the Davis et al. data. Table 1 gives the UCINET partitions for each projection. We then apply these partitions to the rows and columns of the original matrix *A*. The results are shown in Table 2.

The interpretation of the events partitioning is straightforward. As can be seen in Table 2, core events are attended by at least eight women, while peripheral events are attended by at most six women, providing a clear demarcation between core and periphery even without taking into account the core/periphery status of the women attending the events. If we do take into account the status of the women, we find that core events are attended by at least five core women, while peripheral events are attended by at most four core women (and most events are attended by one or two).

In contrast, the core/periphery structure of the women is more subtle. At least one core woman attends only four events, while at least one peripheral woman attends as many as six events. So, sheer number of events does not delineate core from periphery. If we take into account the core/periphery status of the events, we find that a woman is in the core if and only if she attends either four or more core events or at least seven events of all kinds. The complexity of the delineation rule relative to what we could formulate for the events suggests that, whereas events form a clear core/periphery structure, the women do not – something that visualizations of both the bipartite graph and the woman-by-woman valued adjacency matrix tend to confirm.

In this example we used the UCINET procedure which ignores the diagonals but we get exactly the same partition if we use the singular vectors (which would include the diagonals) to obtain the coreness scores and apply the same partitioning method as implemented in UCINET.

7. Dual-projection approach for profile structural equivalence

Two-mode structural equivalence was defined by Borgatti and Everett (1992) and is simply a partitioning of the rows and columns such that the blocks are either all zeros or all ones.[3] They proposed using combinatorial optimization techniques to find structural blockings. However, our interest in this paper is in calculating the degree of structural equivalence between pairs of nodes (Burt, 1976). The degree of structural equivalence is defined as the extent to which two nodes have ties and non-ties to the same others. Interestingly, this is a case where a direct approach formulated on the adjacency matrix of the bipartite graph is less than optimal. For example, suppose we construct the bi-adjacency matrix for the Davis et al. data, and then for each pair of rows we count the number of times they have the same value (either both 1 or both 0). One might expect that counts within-mode would be very high because one thing all women have in common is that they have zeros with all women, bumping up their similarity score. Thus, we would expect women to be more structurally equivalent to women than to events, and vice versa. But the Davis et al. data show that this not necessarily true. In the Davis data, Nora and Evelyn have the same tie value only 60% of the time, while Event 1 and Flora have the same value 83% of the time (see Table 3). By definition, a dual-projection approach avoids such anomalies. This non-confusability of modes is one of the advantages of the dual-projection method.

[3] Borgatti and Everett also extended this definition to valued matrices but we do not pursue that here.

Table 3
Proportion of matching entries in bipartite adjacency matrix representation of Davis data.

		1 EVELY	14 NORA	18 FLORA	19 E1
1	EVELYN	1.000	0.600	0.733	0.700
14	NORA	0.600	1.000	0.800	0.633
18	FLORA	0.733	0.800	1.000	0.833
19	E1	0.700	0.633	0.833	1.000

The dual-projection approach for structural equivalence in two-mode data would be to construct the two projections, calculate similarity matrices for each, run cluster analysis on each, and then map the resulting partitions back onto the two-mode dataset as we did in the core/periphery case. Such an approach yields perfectly acceptable results for the Davis data, but we have reason here to explore a different projection method. We already noted that using AA^{T} and $A^{\mathrm{T}}A$ to build the projections is just one example of a class of possible projections, since all we are doing is constructing similarity matrices based on the rows and columns of the original matrix. Our examination of data loss was dependent on projections constructed in the standard way, and we have not considered this issue on other types of projection. It does, however, seem reasonable to assume that using a dual approach for other projection methods may also have merit and not involve as much data loss as previously thought. We cannot, unfortunately, support this opinion with mathematical evidence in the same way as we did for standard projections. In fact, it is clear this would be an even more difficult mathematical problem that the one already discussed. Nevertheless, we believe it is worth exploring different projections, at least empirically.

Since we are looking for structural equivalence, it may be better to use a projection technique which reflects more closely the structural equivalence measure we intend to utilize. Correlation and Euclidean distance are the most commonly used methods, but as we are using binary data, proportion matching would seem more appropriate. Proportion matching is the number of times two profiles have the same entry (either zeros or ones) divided by the length of the profile. It follows that two structurally equivalent profiles would have a score of 1. We can therefore cluster the projections directly to obtain the structurally equivalent groups. If we had used a projection method that did not reflect structural equivalence then we would have needed to run a profile similarity measure on the projections and then perform a clustering. This extra step would have moved us further from the original data and so may have introduced some additional structural data loss. The proportion matching for the Davis data for the rows and columns is given in Table 4.

We now submit both the women-by-women matrix and the event-by-event matrix to (separate) clustering in order to find the structurally equivalent blocks. There has been general agreement that there are two groups of women in the Davis data (Freeman, 2003), but Doreian et al. (2004) have challenged this and suggested that three groups is a better fit to the data. We clustered the women into two and three groups and found the three groups to be the most interesting. We reproduce that solution here. Our clustering technique is the UCINET (Borgatti et al., 2002) combinatorial optimization routine that searches for clusters such that the average value within groups is as high as possible. Freeman did not come to a conclusion on the number of event groups, but we agree with Doreian et al. that a solution with three groups of events also seems the most appropriate.

The clusterings of the women and the events are then mapped back on to the original data to produce the blockmodel given in Table 5.

In interpreting the results, we first note that this blocking is different from all of the other published blockings for these data, although it bears a general similarity to what Doreian et al. obtained. The Doreian et al. splits were biased to finding null blocks as they set up a heavy penalty for any errors in the null blocks, in addition they were looking for unique solutions. Their (5,5) structural equivalence partition is probably the most consistent with the split found here since merging certain of the women groups can result in a very similar split of the women. However, none of their splits ever separate out event E6 from E7 to E9 as we do here. Direct comparison between the two is difficult as they had very different objectives as well as different methods.

The blocking in Table 5 shows that there is a small core group of events attended by most women (in fact, ten or more). There are two sets of peripheral events which are attended by eight or fewer women. The women are then split into (a) those that attend four or fewer events and consequently attend very few of the peripheral events (Pearl, Ruth, Verne, Myrna, Dorothy, Olivia, Flora), and (b) those that attend core events and the first group of peripheral events (Evelyn, Laura, Theresa, Brenda, Charlotte, Francis, Eleanor), and (c) those that attend core events and the second group of peripheral events (Helen, Katherine, Sylvia, Nora). The number of errors in this solution – that is, the number of ones in zero-blocks and the number of zeros in one-blocks – is 43. This is very close to the minimum possible (40) and similar to the Doreian el al (5,5) structural equivalence split (44) even though our method does not seek to minimize error. What we get is a structure that is consistent with those described by others but one which has a new division of the two peripheries. We can view each of the two peripheral sets of women as having their own local core/periphery structure. Hence, Evelyn, Laura, Theresa, Brenda, Charlotte, Francis, and Eleanor view events E1–E9 as core and E10–E14 as peripheral, whereas Helen, Katherine, Sylvia, and Nora view events E7–E14 as core, with events E1–E6 as peripheral. The remaining women only view the intersection of these two, namely events E7–E9, as core. Hence this model describes the data as two intersecting core/periphery structures, which gives us a new perspective on the data.

8. Dual approach for centrality

If we are to apply the same idea to centrality then we need to find the centrality of each person in the person-by-person matrix and the centrality of the events in the event-by-event matrix. In order to use these scores we need to make the person centralities dependent on the centralities of the events to which they belong and the event centralities dependent on the centralities of the persons that attend that event. This is exactly what Bonacich did in his 1991 paper when he looked at singular vectors to describe simultaneous centralities of actors and groups. As Bonacich pointed out, the singular vectors of A are exactly the same as the eigenvector centrality scores of the bipartite graph representation. In fact we can bring a collection of centrality techniques together by noting that if we use singular vectors to define centrality on a matrix A then (a) if A is the adjacency matrix of a single mode digraph then the centrality scores produced by this method are the hubs and authority centrality scores (Kleinberg, 1999), (b) if A is the adjacency matrix of a single mode undirected graph we simply reproduce the eigenvector centrality scores, and (c) if A is a two-mode matrix, we obtain the Bonacich singular vector measure. In fact, we prove a slightly more general result as follows.

Table 4
Matching projections for the Davis data.

		1 EVE	2 LAU	3 THE	4 BRE	5 CHA	6 FRA	7 ELE	8 PEA	9 RUT	10 VER	11 MYR	12 KAT	13 SYL	14 NOR	15 HEL	16 DOR	17 OLI	18 FLO
1	EVELYN	1.0	0.8	0.9	0.8	0.6	0.7	0.6	0.6	0.6	0.4	0.4	0.3	0.2	0.1	0.2	0.6	0.4	0.4
2	LAURA	0.8	1.0	0.8	0.9	0.6	0.8	0.8	0.6	0.6	0.5	0.4	0.2	0.3	0.2	0.4	0.5	0.4	0.4
3	THERESA	0.9	0.8	1.0	0.8	0.7	0.7	0.7	0.6	0.7	0.6	0.4	0.3	0.4	0.3	0.4	0.6	0.4	0.4
4	BRENDA	0.8	0.9	0.8	1.0	0.8	0.8	0.8	0.6	0.6	0.5	0.4	0.2	0.3	0.2	0.4	0.5	0.4	0.4
5	CHARLOTTE	0.6	0.6	0.7	0.8	1.0	0.7	0.7	0.5	0.7	0.6	0.4	0.3	0.4	0.3	0.5	0.6	0.6	0.6
6	FRANCES	0.7	0.8	0.7	0.8	0.7	1.0	0.9	0.8	0.7	0.6	0.6	0.4	0.4	0.3	0.5	0.7	0.6	0.6
7	ELEANOR	0.6	0.8	0.7	0.8	0.7	0.9	1.0	0.8	0.9	0.7	0.6	0.4	0.5	0.4	0.6	0.7	0.6	0.6
8	PEARL	0.6	0.6	0.6	0.6	0.5	0.8	0.8	1.0	0.8	0.8	0.8	0.6	0.6	0.5	0.6	0.9	0.8	0.8
9	RUTH	0.6	0.6	0.7	0.6	0.7	0.7	0.9	0.8	1.0	0.9	0.7	0.6	0.6	0.4	0.6	0.9	0.7	0.7
10	VERNE	0.4	0.5	0.6	0.5	0.6	0.6	0.7	0.8	0.9	1.0	0.9	0.7	0.8	0.6	0.8	0.9	0.7	0.7
11	MYRNA	0.4	0.4	0.4	0.4	0.4	0.6	0.6	0.8	0.7	0.9	1.0	0.9	0.8	0.6	0.8	0.9	0.7	0.7
12	KATHERINE	0.3	0.2	0.3	0.2	0.3	0.4	0.4	0.6	0.6	0.7	0.9	1.0	0.9	0.7	0.6	0.7	0.6	0.6
13	SYLVIA	0.2	0.3	0.4	0.3	0.4	0.4	0.5	0.6	0.6	0.8	0.8	0.9	1.0	0.8	0.7	0.6	0.5	0.5
14	NORA	0.1	0.2	0.3	0.2	0.3	0.3	0.4	0.5	0.4	0.6	0.6	0.7	0.8	1.0	0.6	0.4	0.6	0.6
15	HELEN	0.2	0.4	0.4	0.4	0.5	0.5	0.6	0.6	0.6	0.8	0.8	0.6	0.7	0.6	1.0	0.6	0.6	0.6
16	DOROTHY	0.6	0.5	0.6	0.5	0.6	0.7	0.7	0.9	0.9	0.9	0.9	0.7	0.6	0.4	0.6	1.0	0.9	0.9
17	OLIVIA	0.4	0.4	0.4	0.4	0.6	0.6	0.6	0.8	0.7	0.7	0.7	0.6	0.5	0.6	0.6	0.9	1.0	1.0
18	FLORA	0.4	0.4	0.4	0.4	0.6	0.6	0.6	0.8	0.7	0.7	0.7	0.6	0.5	0.6	0.6	0.9	1.0	1.0

		1 E1	2 E2	3 E3	4 E4	5 E5	6 E6	7 E7	8 E8	9 E9	10 E10	11 E11	12 E12	13 E13	14 E14
1	E1	1.0	0.9	0.8	0.8	0.7	0.7	0.5	0.4	0.3	0.6	0.6	0.5	0.7	0.7
2	E2	0.9	1.0	0.8	0.8	0.7	0.7	0.5	0.4	0.4	0.6	0.6	0.5	0.7	0.7
3	E3	0.8	0.8	1.0	0.9	0.9	0.8	0.6	0.4	0.2	0.4	0.4	0.3	0.5	0.5
4	E4	0.8	0.8	0.9	1.0	0.8	0.7	0.6	0.3	0.3	0.5	0.6	0.4	0.6	0.6
5	E5	0.7	0.7	0.9	0.8	1.0	0.8	0.7	0.6	0.2	0.3	0.3	0.2	0.4	0.4
6	E6	0.7	0.7	0.8	0.7	0.8	1.0	0.6	0.6	0.3	0.4	0.4	0.3	0.5	0.5
7	E7	0.5	0.5	0.6	0.6	0.7	0.6	1.0	0.6	0.3	0.5	0.4	0.6	0.5	0.5
8	E8	0.4	0.4	0.4	0.3	0.6	0.6	0.6	1.0	0.6	0.4	0.1	0.4	0.3	0.3
9	E9	0.3	0.4	0.2	0.3	0.2	0.3	0.3	0.6	1.0	0.5	0.4	0.6	0.5	0.5
10	E10	0.6	0.6	0.4	0.5	0.3	0.4	0.5	0.4	0.5	1.0	0.7	0.9	0.9	0.9
11	E11	0.6	0.6	0.4	0.6	0.3	0.4	0.4	0.1	0.4	0.7	1.0	0.7	0.7	0.7
12	E12	0.5	0.5	0.3	0.4	0.2	0.3	0.6	0.4	0.6	0.9	0.7	1.0	0.8	0.8
13	E13	0.7	0.7	0.5	0.6	0.4	0.5	0.5	0.3	0.5	0.9	0.7	0.8	1.0	1.0
14	E14	0.7	0.7	0.5	0.6	0.4	0.5	0.5	0.3	0.5	0.9	0.7	0.8	1.0	1.0

Theorem. Let A be a non-square matrix and λ a singular value with corresponding singular vectors x and y derived from AA^{T} and $A^{\mathrm{T}}A$, respectively. Then

(i) If A is symmetric then $x = y$ and x and y are eigenvectors of A.
(ii) If A is square then x and y are hubs and authorities.
(iii) If $B = \begin{pmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{pmatrix}$ then $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x \\ -y \end{pmatrix}$ are eigenvectors of B.

Proof.

(i) Since $A^{\mathrm{T}} = A$ then $AA^{\mathrm{T}} = A^{\mathrm{T}}A = A^2$ so that $x = y$. The result follows from Eq. (1).
(ii) This result follows by definition.
(iii) It is easy to see that if μ is an eigenvalue of B with corresponding partitioned eigenvector $\begin{pmatrix} s \\ t \end{pmatrix}$ then $-\mu$ is an eigenvalue with corresponding eigenvector $\begin{pmatrix} s \\ -t \end{pmatrix}$. Furthermore

$$\begin{pmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{pmatrix} = \mu \begin{pmatrix} s \\ t \end{pmatrix}$$

so that $At = \mu s$ and $A^{\mathrm{T}}s = \mu t$ hence

$$A^{\mathrm{T}}At = \mu A^{\mathrm{T}}s = \mu^2 t \quad \text{and} \quad AA^{\mathrm{T}}s = \mu At = \mu^2 s.$$

Therefore if μ is positive it is a singular value and s and t are corresponding singular vectors. Given the structure of the eigenvalues of B, the result follows.

We note that it may be possible to use other centrality measures, but they would need to be ones that are applicable to valued data and there would need to be a connection between the centralities of the persons and the events. Degree, for example, could be used in this way, but the results would be very similar to the singular vectors. The reason for this is they could be viewed as early iterations of the power method which can be used to find eigenvectors and hence singular vectors.

Table 5
Structurally equivalent blocking of the Davis data based on clustering of projections derived from matching.

		11 E	12 E	10 E	14 E	13 E	9 E	8 E	7 E	2 E	3 E	1 E	5 E	6 E	4 E
1	EVELYN						1	1		1	1	1	1	1	1
2	LAURA							1	1	1	1	1	1	1	
3	THERESA						1	1	1	1	1		1	1	1
4	BRENDA							1	1		1	1	1	1	1
5	CHARLOTTE								1		1		1		1
6	FRANCES							1			1		1	1	
7	ELEANOR							1	1				1	1	
8	PEARL						1	1						1	
9	RUTH						1	1	1				1		
10	VERNE		1				1	1	1						
11	MYRNA		1	1			1	1							
16	DOROTHY						1	1							
17	OLIVIA	1					1								
18	FLORA	1					1								
15	HELEN	1	1	1				1	1						
12	KATHERINE		1	1	1	1	1	1							
13	SYLVIA		1	1	1	1	1	1	1						
14	NORA	1	1	1	1	1	1		1					1	

9. Conclusions

Researchers often avoid analyzing projections on the assumption that they do not contain all of the structural information in the original two-mode matrix. In this paper we have critically examined this assumption and argued that in most of the cases we are likely to encounter in empirical settings, the projections are complete, meaning that together they contain all the information needed to reconstruct the original two-mode matrix. As a result, we feel that dual-projection analysis methods are generally safe to use and often have conceptual advantages over direct methods (and in many cases can be used to obtain the same results as direct methods). While it is clear these methods have to be assessed more fully than we have done here, it also seems clear that the dual-projection approach has more potential than previously thought.

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