

1 Introduction

Many classic network datasets include both positive and negative relations. For example, among the standard datasets included in UCINET (Borgatti, Everett and Freeman, 2002), the Sampson monastery data, Bank Wiring Room and Read's highland tribes all have negative relations. Negative relations are fundamental to certain theoretical approaches in network analysis, such as balance theory (Heider 1946, Cartwright and Harary 1956), and the related clusterability theory (Davis 1967). In addition, negative relations have been a standard part of blockmodeling (Lorrain and White, 1971; Breiger, Boorman and Arabie 1975, Everett and Borgatti 1995) and semigroup work (Boyd, 1990). Furthermore, there is a considerable psychological literature on negative ties (Taylor, 1991) and conflict (Tajfel and Turner, 1979). Recently, negative interactions like bullying and social exclusion have been the subject of extensive research (DeWall, 2013).

We note that our interest is specifically in relations that are in themselves negative, rather than positive relations that may have negative consequences. For example, positive relations may enable the flow of useful ideas and emotional support, but may also transmit disease and misinformation. Thus, this paper is not intended as a contribution to the 'dark side of social capital' literature (Portes and Sensenbrenner, 1993; Gargiulo and Benassi, 2000). Rather, we are concerned with directly negative relations, such as the antagonistic "hina" relation reported by Read (1954), the conflict relation in the "bank wiring room" data reported by Roethlisberger and Dickson (1939), and the judgments of dislike and disesteem among monks reported by Sampson (1969). All of these represent negative sentiments or behaviors towards other actors in the network.

The question we address in this paper is how to analyze negative tie data. One reason for concern is that negative relations tend to form different structures than positive relations do. For example, in positive tie networks we almost always see high levels of transitivity – e.g., the friends of friends are often friends. But in negative tie networks we see very low levels of transitivity: enemies of enemies do not tend to be enemies. As a result, they tend not to have any clustering. Even more fundamentally, negative tie networks tend to be very sparse, making it difficult to fit tie-level models, and typically resulting in highly disconnected networks, making it impossible to apply certain network analysis methods.

A deeper reason for concern about negative tie networks is that social processes that occur in positive tie networks may not occur in negative tie networks. For example, in a friendship network, we expect the ties to serve as a backcloth along which traffic may flow (Atkins, 1977). So if there is a directed path from A to B to C to D, we expect that over time and with some probability something (e.g., information) could diffuse from A to D. Much of the machinery of network analysis, especially centrality, is based on this expectation (Borgatti, 2005). But, under what circumstances can we expect flows beyond the dyad in a negative tie network? Suppose A dislikes B, and provides B with some embarrassing news about B's mother. If B dislikes C, can we really expect B to pass the original message along to C?

In this paper we present a general assessment of which network analytic concepts and techniques apply to negative relations, and how interpretations need to be adjusted when they are applied to negative ties. We also introduce new concepts and measures specifically designed for negative ties.

In so doing we hope to provide tools that will help network researchers understand networks of negative ties or at least provide them with tools that will enable them to test empirical hypothesis about such networks. For example at the node level if we had robust negative tie centrality measures we could examine whether people with high or low centrality are ignored or preferred in who is chosen for promotion in an organizational trust network. At the network level if we had measures that accommodate both positive and negative ties we could see the effect of negative ties on the centralities of all the actors in the network and hence determine how detrimental (or not) negative ties are. We may be able to detect potential victims or groups of victims in a bullying network and hence plan an intervention at an early stage which provides support and so prevents an escalation. We do not specifically aim to demonstrate the full potential of the methods we propose here, but hope to start to build a collection of tools that will aid empirical analysis.

2 Standard Methods

There is one class of standard network concepts that clearly applies to negative data without modification of any kind. This is the set of positional or role equivalence concepts, such as structural equivalence (Lorrain and White, 1973), automorphic equivalence and regular equivalence (White and Reitz, 1983; Everett and Borgatti, 1995), all of which are indifferent to the type of relation they are applied to. For example, if nodes A and B are structurally equivalent in a directed negative-tie network, it means that A and B send negative ties to the same third parties, and receive negative ties from the same parties. Structurally equivalent nodes are typically expected to have similar outcomes with respect to structural processes, and this applies to negative ties as well. Structurally equivalent nodes are also seen as occupying similar positions or playing the same roles in a network, and again this will be the same for negative ties. The same applies to other equivalences.

In principle, statistical techniques such as QAP correlation (Hubert and Schultz, 1976) and regression (Dekker, Krackhardt and Snijders 2007) can be applied to both positive and negative directed tie data, although the specific models we fit may be different, and not just in a mirror-image way. For example, sameness of language might be positively related to positive ties, but wouldn't necessarily be negatively related to negative ties – after all, sometimes negative ties arise precisely because people are able to communicate with each other. In practice, the sparseness of negative tie data can sometimes cause problems with estimation. Similar considerations apply to exponential random graph models (ERGM). The overall framework is perfectly applicable to negative data but the models that actually fit are likely to be different. Moreover, it may be that many of the configurations (both directed and undirected) currently available in ERGM software packages are less relevant for negative ties, and new configurations should be developed.

The situation with centrality measures is a bit more complicated. Perhaps the most translatable centrality measure is simple degree (Freeman, 1979). In certain respects, degree makes perfect sense for negative ties. For example, if the directed relation is “dislikes”, then the actors with high indegree can be described as the most disliked in the network -- wholly parallel to the case of a “likes” network, where indegree indicates popularity. Similarly, in the context of social capital, high degree in a positive-tie network represents an asset in an actor's social ledger (Labianca and Brass, 2006), while high degree in a negative-tie network represents a liability. The difference in interpretation between degree in positive and negative tie networks parallels the difference in the interpretation of the ties themselves, which causes

us no difficulties. Bonacich and Lloyd (2004) extend eigenvector centrality to networks of negative ties resulting in a status measure. We explore their ideas more fully later in this paper.

On the other hand, one way that we commonly interpret degree centrality is in terms of risk of exposure to something flowing through a network (Borgatti, 2005). For example, suppose a virus enters a group at a random node and is transmitted at random to an adjacent node, and so on. The probability of a random walk reaching a particular node after a given number of hops is a function of the degree of that node. Hence, in a positive-tie network, degree centrality provides an index of exposure. However, as noted earlier, in a negative-tie network we don't normally expect things to flow along paths of length greater than one, in which case degree centrality will not function as an index of exposure.

Similar considerations apply to other degree-related concepts, such as graph density and degree-based graph centralization.¹ For example, for positive ties, increased density would suggest a group with greater social cohesion. For negative ties, we would generally expect the opposite. However, there are cases where the parallel is less exact. For example, if in the negative-tie case all ties involve just a few individuals (i.e., high graph centralization or a core/periphery structure), the overall effect on the network could be an increase in cohesion as the majority of the group bonds over the common enemy.

Whereas degree centrality can be interpretable in settings where flows do not make sense, betweenness centrality is difficult to interpret in the absence of flows. As Freeman (1979) defines it, a node's betweenness is essentially the number of times that the node is along the shortest path between two others. Where there are multiple shortest paths between two nodes, the focal node's betweenness score is incremented by the proportion of those shortest paths that it is along. As discussed in Borgatti (2005), the formula for betweenness centrality gives the expected number of times that something flowing through a network (either directed or undirected) will reach a particular node, given that it travels only on shortest paths and chooses at random between equally short paths. It is difficult to see how this measure could be interpreted in the absence of some kind of a flow process. This is especially true for flow betweenness (Freeman, Borgatti and White, 1991), which is built on the concept of maximum flow through a system of pipes whose capacities are given by the strengths of ties. Most variants of betweenness centrality depend on the notion of flow, and as such are generally inappropriate for negative-tie networks.

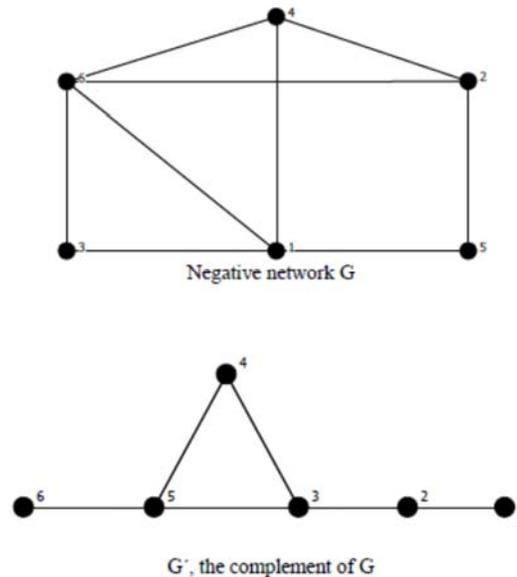
Closeness centralities (Freeman, 1979; Valente & Forman, 1998; Stephenson & Zelen, 1989) are also difficult to interpret in the absence of flows. All closeness centralities summarize the length of paths (or, more generally, directed walks) that link a node to the rest of the network. The longer these paths, the greater the amount it takes for things to flow between the nodes, and the greater the probability of a failure to flow. Therefore, nodes separated from the network as a whole by longer paths are disadvantaged relative to those with short paths with respect to certainty and time until arrival of things flowing through the network. Without something flowing, it is hard to see why we would invoke the concept of path length.

3 Graph Complement Approach

¹ Our remarks here and in subsequent paragraphs apply equally to directed and undirected graphs.

For techniques that only make sense for positive data, one obvious approach is to transform the negative data into positive data by taking the graph complement. If G is a graph (or digraph), then we can define the complement G' as a new graph with the same vertices as G and where two vertices are adjacent if and only if they are not adjacent in G . Figure 1 shows an example of a graph and its complement, with the nodes rearranged to give a better visualization.

<INSERT FIGURE 1 ABOUT HERE>



Once we have the complement, the idea is to apply all of our standard network analysis techniques, as if it represented positive ties. However, treating the complement as positive ties makes a strong claim. The claim is justified when the relation only has two states, and not-State 1 is necessarily State 2. For example, in a roster-based network survey where we ask respondents to check off the names of people they know, the lack of a check means they don't know the person. However, in the case of affective ties, there are more possibilities. One can like someone, dislike them or simply not know them well enough to either like or dislike them. Hence the complement of 'like' includes both 'does not like' and 'has no relationship with' and this is not quite the same as 'dislike'. Empirically, the complement is the most meaningful when everyone knows everyone else very well. For example, in a small work team such as a military squad, the complement of an affective relation such as distrust could well be interpretable as trust.

Even when the complement is interpretable, however, there remain other issues. Obviously, the structure of the complement of a graph is, in general, different from the structure of the original graph. For example, a well-known property of graph complements is that both G and G' cannot be disconnected. If an observed negative tie network is connected, the complement may not be, and methods like closeness centrality will be inappropriate. Similarly, the complement of a graph with low density – which most real graphs have – will have high density. Unfortunately, high density graphs are often not very interesting to analyze as they have very little subgroup clustering and many measures of centrality will show very little variance.

In some cases, though, the analytical method is indifferent to the structural differences between a graph and its complement. For example it is clear that if vertex x has degree ρ in G and G has n vertices then vertex x has degree $n-1-\rho$ in G' . So a node's degree in G' is predictable from its degree in G (this applies to in/out degree as well). In contrast, we could not deduce, say, the value of a node's betweenness centrality in G' , given its betweenness in G . The same is true of eigenvector centrality (except in the case of a regular graph). We shall call methods whose results in G' can be deduced directly from the results in G *complement-consistent* methods.

Other examples of complement-consistent concepts are structural equivalence and automorphic equivalence: all structurally or automorphically equivalent actors in G are equivalent in G' . In general regular equivalences are not complement-consistent.

In general, no measures of centrality are complement-consistent, with the previously-mentioned exception of degree centrality. Exponential random graph models are in principle complement-consistent if complement configurations are used, although in practice the excessive density of negative-tie complement networks may make estimation impossible.

In sum, there are certain concepts and methods that yield the same results for both a graph and its complement. Not surprisingly, these are the same concepts and methods discussed in the previous section as approaches that are interpretable in both positive and negative tie networks. For methods that are not readily interpretable on negative tie data, we have the option of analyzing the complement, but as noted above, there are problems here too. What is needed is a set of methods specifically tailored for negative data. We make a tentative start in this direction in the following sections.

4 Negative centrality

If we have a negative tie network, we prefer our centrality measures to be comparable with centrality in a positive-tie network. That is, we want low values to represent actors who are not very central and high values for those who are central. This is precisely what happens if we use the graph complement approach with degree. That is, negative degree centrality is simply the degree of the node in the complement graph – i.e., $n-1-d(x)$, where $d(x)$ indicates the degree of node x . As with ordinary degree centrality, this can be normalized by dividing through by $n-1$. We denote normalized negative degree centrality of actor x by $d^*(x)$, as shown in Equation 1. With this measure, someone who is disliked² by a lot of people will receive a low centrality score and someone who is not disliked at all will receive a high score.

$$d^*(x) = 1 - \frac{\rho(x)}{n-1} \quad (1)$$

Note that this equation weights everyone the same, but in many cases we might prefer a measure in which actors with high centrality affect the centrality score of the actors they are connected to more than those with a low score. In other words, we might want to capture the notion that to have negative connections with actors who have a low centrality score is better than having negative connections to actors who have a high score. This is similar in spirit to eigenvector centrality, in which the centrality of a node is proportional to the centrality of the

² For convenience, we use “dislike” as a generic example of a negative tie.

nodes it is connected to, although in our case the result is not an eigenvector of a particular matrix. Taking this approach, we denote the centrality of x by $h^*(x)$ and obtain Equation 2. In the equation, $N(x)$ indicates the neighborhood of x , which is to say, the set of nodes adjacent to x .

$$h^*(x) = 1 - \sum_{y \in N(x)} \frac{h^*(y)}{n-1} \quad (2)$$

We can represent this in matrix form as

$$\mathbf{h}^* = \left(\mathbf{I} + \frac{1}{n-1} \mathbf{A} \right)^{-1} \mathbf{1} \quad (3)$$

where \mathbf{I} is the identity matrix, \mathbf{A} is the adjacency matrix transformed, and $\mathbf{1}$ is the vector of all ones.

Applied to positive symmetric adjacency matrices, Equation 3 is similar to the centrality measures proposed by Katz (1953), Hubbell (1965), and Bonacich (1987). Hubbell's measure is given by Equation 4, and we can see that h^* is precisely Hubbell's measure with a fixed β of $-(n-1)^{-1}$. Katz's measure is just Hubbell's minus one, using the same β . Bonacich's measure is Katz's divided by β (see the Appendix for derivations).

$$\mathbf{h} = (\mathbf{I} - \beta \mathbf{A})^{-1} \mathbf{1} \quad (4)$$

Neither Katz nor Hubbell discussed a negative beta, but Bonacich suggested that a negative beta would be useful for modeling competitive situations where it is better for a node to engage with weak alters than with strong ones. When it is beneficial to an actor to be connected to well-connected others – as envisioned by Katz and Hubbell – we use a positive β as usual. But when it is harmful to an actor to have ties to well-connected others, we use a negative β . In proposing the h^* measure, we are asserting that the same logic that applies to competitive situations applies to negative ties, and therefore fix a negative β .

Table 1 gives the values of negative degree centrality d^* and our h^* measure for the “hina” relation collected by Read (1954). This dataset is available in UCINET (Borgatti et al 2002), where it is called GAMANEG.

<INSERT TABLE 1 ABOUT HERE>

	d^*	h^*
1 GAVEV	66.67	71.83
2 KOTUN	66.67	72.46
3 OVE	86.67	90.38
4 ALIKA	93.33	95.21
5 NAGAM	73.33	81.31
6 GAHUK	66.67	75.05
7 MASIL	100.00	100.00
8 UKUDZ	93.33	94.41
9 NOTOH	73.33	80.69
10 KOHIK	80.00	85.69
11 GEHAM	66.67	74.44

12	ASARO	73.33	80.56
13	UHETO	73.33	80.97
14	SEUVE	80.00	83.78
15	NAGAD	60.00	67.76
16	GAMA	60.00	68.26

Table 1. d^* and h^* scores for GAMANEG

Both methods, of course, give maximum centrality to the isolate, Masil, but, usefully, h^* differentiates between actors with the same d^* score. For example, Gavev, Kotun, Gahuk and Gehan all have the same d^* score, but have different h^* scores. Hence, in this instance, we see h^* as a refinement of d^* , much as we might regard eigenvector centrality as a refinement of degree.

Solving equation 3 is standard and can be achieved exactly in time complexity $O(n^3)$ and so presents little computational difficulty for modestly sized networks consisting of thousands of nodes. For larger networks we can use iterative methods based on equation 2 (not as accurate but extremely fast) and so there are no real restrictions from a computational point on the size of the network.

For directed graphs, it is a simple matter to extend d^* to negative in-degree and negative out-degree versions denoted by d_{in}^* and d_{out}^* respectively. A high d_{in}^* score identifies an actor who is in a good position with regard to incoming negative ties, meaning they have few of them. A high d_{out}^* score indicates an actor who dislikes few others, and may therefore be free of certain stresses.

To extend the in/out distinction to h^* , we need to take into account the scores of the nodes that a node is connected to. Since these are negative ties, we want an actor with a low h_{in}^* score to be someone who has many incoming negative ties from people who tend not to give a lot of negative ties, which is to say have a high h_{out}^* score. The logic is that receiving negative ties is most problematic when the ties are from discriminating people who dislike very few others. Ties from those that dislike everyone do not harm an actor, and indeed may serve as a badge of honor. An actor with high h_{in}^* would be someone who receives very few negative ties from discriminating others. If they do receive negative ties, it is from people who dislike everyone, which makes the ties less significant. Similarly, we want an actor with a low h_{out}^* to be someone who sends negative ties to many people who tend not to receive a lot of negative ties – i.e., those with high h_{in}^* scores. An actor with low h_{out}^* is out of sync with the rest of the network, disliking those that others like. An actor with high h_{out}^* has few outgoing negative ties, and those they have tend to be to people who receive many negative ties – people with low h_{in}^* . Thus, we define the directed versions of h^* , h_{out}^* and h_{in}^* as shown in Equations 5 and 6,

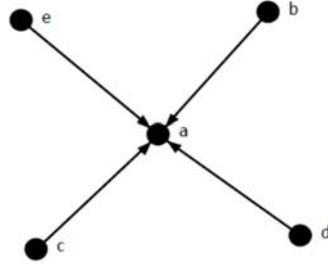
$$h_{out}^*(x) = 1 - \sum_{y \in N_o(x)} \frac{h_{in}^*(y)}{n-1} \quad (5)$$

$$h_{in}^*(x) = 1 - \sum_{y \in N_i(x)} \frac{h_{out}^*(y)}{n-1} \quad (6)$$

where $N_i(x)$ and $N_o(x)$ are the in-neighborhood and out-neighborhood of node x , respectively. The equations are similar in spirit to the definitions of authorities and hubs (Kleinberg, 1999) in as much as the “out” portion of the measure is related to the “in” portion of the members of

the out-neighbourhood and the “in” portion to the “out” portion of the in-neighbourhood. However as in the undirected case the result is not an eigenvector type measure.

<INSERT FIGURE 2 ABOUT HERE>



Node	h_{in}^*	h_{out}^*
a	0.000	1.000
b	1.000	0.996
c	1.000	0.996
d	1.000	0.996
e	1.000	0.996

Table 2. h^* scores for nodes in Figure 2

Figure 2 illustrates the calculation of h_{out}^* and h_{in}^* (results shown in Table 2). The center node a receives negative ties from all nodes, and so has the lowest possible h_{in}^* centrality of zero. None of the other nodes receives any negative ties, so they have the largest possible h_{in}^* centrality. With respect to h_{out}^* , the center node sends no outgoing ties, so has perfect h_{out}^* centrality. The other nodes suffer from each having an outgoing negative tie, but not very much because the node they dislike is the one that everybody dislikes.

<INSERT TABLE 2 ABOUT HERE>

The converged values h_{out}^* and h_{in}^* can also be formulated in matrix terms,

$$h_{out}^* = \left(I - \frac{1}{(n-1)^2} AA^T \right)^{-1} \left(I - \frac{1}{n-1} A \right) \mathbf{1} \quad (7)$$

$$h_{in}^* = \left(I - \frac{1}{(n-1)^2} A^T A \right)^{-1} \left(I - \frac{1}{n-1} A^T \right) \mathbf{1} \quad (8)$$

where A^T = transpose of A , and I and $\mathbf{1}$ are as before.

We illustrate the directed version of these measures on the Sampson (1969) data given in UCINET. We have obtained an overall negative dataset by aggregating all the negative relations (Dislike, Disesteem, Negative Influence and Blame) and then dichotomizing the result. The results are given in Table 3, where columns 1 and 2 give the d^* scores and columns 3 and 4 give the h^* scores.

<INSERT TABLE 3 ABOUT HERE>

		d_{in}^*	d_{out}^*	h_{in}^*	h_{out}^*
1	ROMUL_10	88.24	100.00	91.27	100.00
2	BONAVEN_5	100.00	82.35	100.00	91.56
3	AMBROSE_9	100.00	70.59	100.00	82.48
4	BERTH_6	70.59	58.82	76.25	74.40
5	PETER_4	52.94	76.47	62.05	81.14
6	LOUIS_11	70.59	52.94	77.52	67.80
7	VICTOR_8	64.71	52.94	71.61	68.95
8	WINF_12	82.35	82.35	86.51	89.66
9	JOHN_1	88.24	58.82	91.24	70.66
10	GREG_2	58.82	76.47	69.46	85.50
11	HUGH_14	70.59	58.82	77.30	75.14
12	BONI_15	100.00	64.71	100.00	79.36
13	MARK_7	76.47	70.59	82.71	82.13
14	ALBERT_16	76.47	64.71	83.19	79.36
15	AMAND_13	58.82	64.71	67.93	77.90
16	BASIL_3	29.41	70.59	44.99	78.34
17	ELIAS_17	35.29	82.35	48.90	87.65
18	SIMP_18	35.29	70.59	49.57	77.72

Table 3 Sampson negative-tie data with directed d^* and h^* scores

As in the undirected case, we see that actors with the same score for the d^* measure are nearly always separated for the h^* measure (the exceptions are those with the maximum score of 100 and two of the three actors with a d_{out}^* score of 64.71). We also note that, for d_{in}^* , Basil gets a score far lower than any other actor, but for h_{in}^* his score is more similar to the other ‘outsiders,’ Elias and Simplicius. This is because although Basil receives a lot of negative ties, they are from actors not as positively thought as some others, raising his score relative to simple degree.

So far we have only looked at binary data. Clearly, we can calculate d^* for valued data, although the normalization no longer yields values between 0 and 1, and we have to interpret values in the same way as we do when extending degree centrality to valued data. Similarly, the h^* measure will also work for valued data, provided the largest eigenvalue of the adjacency matrix is less than $n-1$.

5 Negative cohesive subgroups

We begin our discussion of cohesive subgroups on negative ties with a discussion of perhaps the most fundamental of subgroup concepts, the clique. The standard clique definition is a maximal cohesive subgraph (Luce, 1949), meaning a group in which everyone is connected to everyone else within the group and the group cannot be made larger without violating this

condition. Obviously, if we apply cliques to negative data, we get maximal clusters of nodes that mutually dislike each other. These clusters are by definition not cohesive, but neither necessarily are the complements of the cliques. How then to define cohesive subgroups for negative ties?

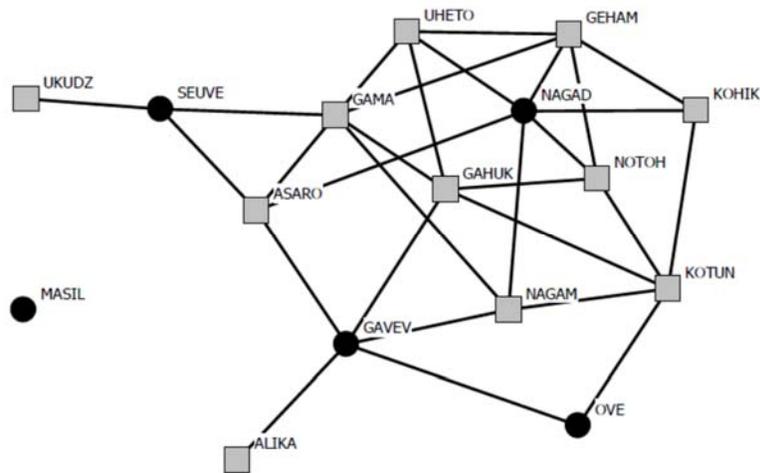
First, we note that the clique concept is concerned with relations within the group. But with negative ties, we suggest that we reverse this and look at relations outside the group. Second, cliques are defined to be maximal, but in dealing with negative data, it would make more sense to define them as minimal. Consequently we define *negative cliques* as minimal subgraphs in which each vertex outside the subgraph has a connection to a vertex in the subgraph. A negative clique is a smallest group in which everyone outside the group dislikes someone within the group.

To illustrate the concept, we use the GAMANEG data previously analyzed for negative centrality. The network has 6 negative cliques, namely

- Gavev Ove Masil Seuve Nagad
- Gavev Kotun Masil Seuve Nagad
- Gavev Kotun Masil Ukudz Uhetu
- Gavev Kotun Masil Geham Seuve
- Gavev Kotun Masil Uhetu Seuve
- Kotun Alika Masil Uhetu Seuve

The first of these groups is shown in Figure 3, with the members of the negative clique shown in black circles and the rest of the actors in grey squares. We note that Masil is a member every negative clique. This is always true for isolates -- it is a consequence of the negative clique definition. In practice it may be convenient to remove the isolates (we could of course modify the definition to exclude isolates), but we have not done so here.

<INSERT FIGURE 3 ABOUT HERE>



What defines the negative clique is that every light, square actor has a connection to a dark, circle actor. Negative cliques can overlap and to manage this we can employ all of the standard methods for dealing with ordinary clique overlaps. We can also, like cliques, specify a maximum size (analogous to the minimum size for cliques). We can also generalize negative cliques in the same way that the clique concept has been generalized. For example we can define a negative k -plex as a minimal group in which all but k actors outside the group have a connection to an actor within the group. We could also use negative cliques to define new centrality measures by simply counting the number of negative cliques an actor is in, possibly weighted by size (Everett and Borgatti, 1998).

As an area of application for negative cliques we could think of an organizational restructuring. The re-assignment of a negative clique to another location would reduce tension in the remaining network because everyone would have at least one person they disliked removed. Overall this could result in a positive outcome for all those that remain.

Negative cliques are a well-known concept in graph theory where they are known as minimal dominating sets. This means that there are a number of theorems and results which may prove useful. In addition algorithms have been developed to find minimal dominating sets, although (like cliques) the related decision problem is known to belong to the class NP.

Unlike the clique, the concept of a negative clique has even more meaning for directed data, as in this case a negative clique can be defined as a minimal group that receives negative ties from all actors outside the group. This preserves the idea that it is the smallest group in which every actor outside the group dislikes at least one of the group members. However the converse may also be of interest, i.e., the smallest group that sends ties to every actor outside the group. Such a group may be seen as undesirable and consist of actors that cause disruption.

Using the negative-tie Sampson data defined previously, an example of a directed negative clique would be Peter, Romuald and Elias. Romuald has to be included as he has no outgoing ties. There are two other negative cliques, namely {Peter, Romuald, Simplicus} and {Romuald, Greg, Elias}.

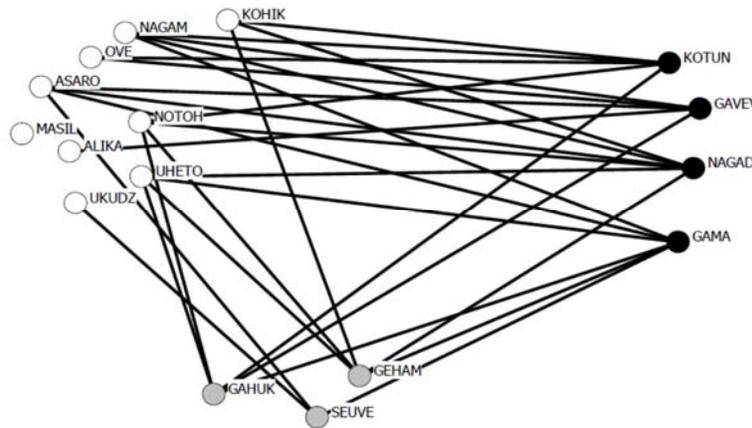
It should also be noted that negative cliques are closely related to the concept of keyplayers (Borgatti 2006) since they are a group who can reach everyone in the network and are

therefore a solution to the keyplayer positive problem. This means the keyplayer program can be used to help find negative cliques, although it is not set up to find all the negative cliques and will terminate when it has found just one.

Whereas cliques take into account only ties within the group (and their counterparts, negative cliques, are only concerned with ties from nodes outside the group) the concept of factions simultaneously requires maximum density within groups and minimum density between groups. Thus, a perfect faction solution consists of complete components. To apply this concept to negative data we simply define negative factions in terms of a partition of nodes such that the density of ties within groups is minimized and the density of ties between groups is maximized. This is available in UCINET's combinatorial clustering routine by specifying that ties are "dissimilarities" or negative rather than "similarities" or positive.

Figure 4 shows the results of applying this algorithm to the GAMANEG dataset. In this case, a 3-cluster solution was chosen. It can be seen that all negative ties are between groups, and no negative ties exist within groups.

<INSERT FIGURE 4 ABOUT HERE>



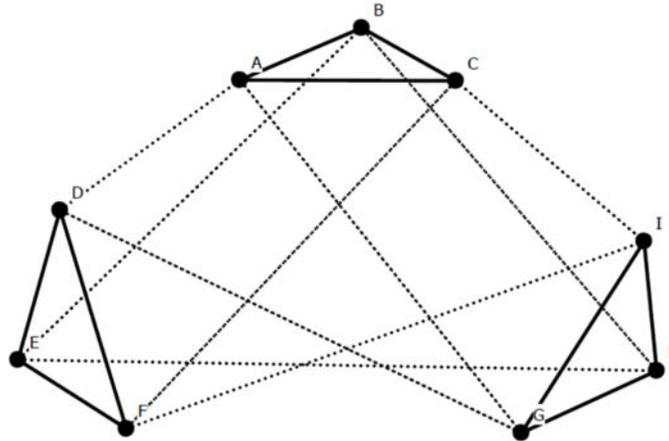
6 Mixed data and PN centrality

So far we have only considered data that consists entirely of negative ties. If we have a mixture of positive and negative ties then we could analyze the positive and negative relations separately and combine the results in some way. However it would seem preferable to give joint consideration to both types of tie. Balance theory and clusterability and their generalizations (Doreian and Mrvar 2009) have to a certain extent addressed this issue with respect to cohesive subgroups and we offer no new developments in that area. We shall therefore concern ourselves with centrality type measures.

When we have mixed data it is common practice to represent negative ties with negative values. This has been the approach taken by Bonacich and Lloyd (2004) in their status paper. A consequence of this, which they acknowledge, is that any method which involves products of edges (even indirectly) has an underlying model of structural balance. They propose using eigenvectors as a measure of status in an undirected network with both positive and negative ties. One problem they do not address is that, with negative values, the matrix may not have a dominant eigenvalue. This means it is not clear which eigenvector we should use. In addition

it is possible for the adjacency matrix to have repeated eigenvalues and hence multiple linearly independent eigenvectors. In this case certain centralities can be arbitrarily assigned. As an extreme example consider the network shown in Figure 5, in which each actor has two positive ties (shown as thick black lines) and two negative ties (shown as thinner grey lines).

<INSERT FIGURE 5 ABOUT HERE>



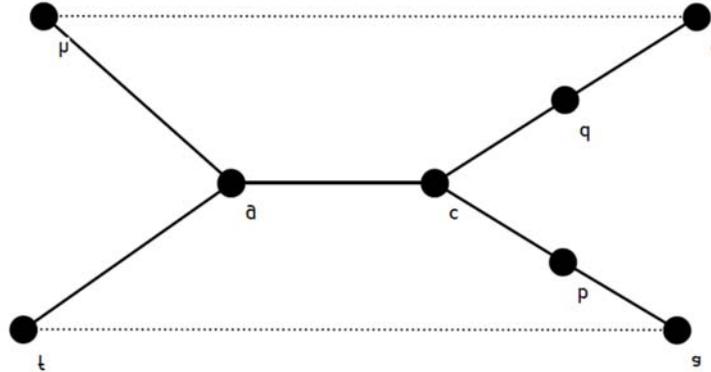
The graph is clusterable but not balanced, and in fact has quite a few unbalanced triads. The graph is also transitive, meaning that every node is automorphically equivalent and so we would expect any centrality measure to give each actor the same score. In this case the vector of all ones will be an eigenvector. The eigenvalues are -3, 0 and 3 and so we do not have a dominant eigenvalue. The eigenvalues of 3 and -3 both have two independent eigenvectors but no linear combination of these can be found that produces the vector of ones. The eigenvalue of 0 has five independent eigenvectors and the space spanned by these does include the one vector. Hence if we wanted to select a status score we would have needed to have picked eigenvectors corresponding to zero and even then it would not be clear which one to select. This example shows that not only is it clear that we do not know which eigenvalue to pick, that even if we pick the right one we still may have a large dimensional space from which to pick our eigenvector and hence our status score. This really means that at a practical level as a centrality or status type measure this has limited value. If in addition our data was directed then there may not even be any real eigenvalues or eigenvectors.

Recently, Smith et al (2013) proposed a new measure, called PII, that can handle a mix of positive and negative ties. However, the measure is intended for a very specific context in which actors achieve power by avoiding dependency on others. In this measure, having even positive ties with well-connected others is detrimental to an actor's power. For example, for the positive-tie network shown in Figure 6, PII gives a low score to node c. Similarly, PII also gives a very low score to node c in the mixed network shown in Figure 7.

<INSERT FIGURE 6 ABOUT HERE>



<INSERT FIGURE 7 ABOUT HERE>



In our case, we are looking to generalize traditional measures in which having positive ties to well-connected others contributes positively to a node’s centrality. One approach is to use our h^* measure for the negative ties, and then, since it is so similar to Hubbell’s measure, use Hubbell or something very much like it for the positive ties. But this still leaves the issue of choosing β for the positive case. We could use the same value of β used in h^* , namely $1/(n-1)$ but this value was chosen to normalize the negative tie scores. It would therefore seem to make more sense to pick a value of β that normalizes the positive ties scores. The maximum centrality scores for Hubbell occur when we have a complete network and to normalize these values we need to select a value of β of $1/(2n-2)$. Since we need to look at the interaction of both the positive and the negative ties we can achieve this by using a β value of $1/(2n-2)$ and taking twice the negative ties from the positive ties and applying equation 4. We call this result PN centrality and formally this is given by equation 9.

$$PN = \left(I - \frac{1}{2n-2} A \right)^{-1} \mathbf{1} \tag{9}$$

In the equation, $A = P - 2N$, where P denotes a positive tie matrix and N denotes a negative tie matrix.

To illustrate the measure, we again look at the Sampson monastery data in which we have dichotomised and symmetrized all the positive and negative ties at time period three. Table 4 gives the results of using degree, status, PII and PN.

<INSERT TABLE 4 ABOUT HERE>

		Degree	Status	PN	PII
		-----	-----	-----	-----
1	ROMUALD	2	0.155	1.030	0.01
2	BONAVENTURE	6	0.325	1.157	1.95
3	AMBROSE	4	0.339	1.074	1.61
4	BERTHOLD	-1	0.295	0.897	0.65
5	PETER	-3	0.299	0.775	1.18
6	LOUIS	-1	0.297	0.812	-0.55
7	VICTOR	-3	0.260	0.763	-0.18
8	WINFRID	1	0.088	0.942	0.76
9	JOHN_BOSCO	3	0.000	0.900	0.38
10	GREGORY	5	0.014	0.993	1.09
11	HUGH	-2	0.097	0.853	0.09
12	BONIFACE	0	0.099	0.952	0.69
13	MARK	-1	-0.027	0.860	0.00
14	ALBERT	-2	0.103	0.855	-0.51
15	AMAND	-4	-0.067	0.645	-0.20
16	BASIL	-9	-0.345	0.393	-0.30
17	ELIAS	-7	-0.351	0.476	-1.16
18	SIMPLICIUS	-8	-0.371	0.415	-0.68

Table 4 Degree, Status, PN and PII on the mixed data.

We can see from column one that degree identifies the group of outsiders (Basil, Elias and Simplicius) as does PN where they have significantly lower scores. Status identifies them together as they all have a high score and they all have the same sign. It is not possible however to view them as outsiders as we do not know if their high score is derived from negative ties to the positively signed actors or positive ties to the negatively signed actors, although we could look back at the original data to see if this is the case. Recall that the status score assumes the data can be partitioned into two groups, and as we have already shown can have difficulty if there are more than two groups resulting in anomalous scores. We see that Gregory has a high degree centrality score and was in fact seen as a leader of the Young Turks (Winifrid, John Bosco, Gregory, Hugh, Boniface, Mark and Albert) by Sampson. Status however gives Gregory a low score (0.014) and fails to identify him as a central actor. The PII measure does give Gregory a high score but it fails to identify the outsiders (Louis and Albert both have scores that are between Basil and Simplicius). We need to acknowledge that PII does not claim to be a normal centrality type measure and so this does not mean this measure is limited, but it does mean care needs to be taken in interpreting the results. Our point is that PN and degree in this instance are the only two measures that capture our intuitive idea of what a centrality measure is. One issue with degree is that it is not a very sensitive measure but this is not true of PN. Berthold, Louis and Mark all have the same degree but have different PN scores. These scores range from 0.81 to 0.90 and in addition we note that Albert has a score of 0.87 showing that the rank ordering also changes when we use PN.

The following example highlights the advantages PN has over straight degree. Consider the network whose adjacency matrix is shown in Table 5. In this network, the row and column

sums are all zero and so degree centrality gives every actor the same score. PN centrality on the other hand gives values that range from 0.84 to 0.91 as shown in Table 6.

<INSERT TABLE 5 ABOUT HERE>

		1	2	3	4	5	6	7	8	9	10
	a	b	c	d	e	f	g	h	i	j	
1	a	0	1	0	1	0	0	0	-1	-1	0
2	b	1	0	1	-1	1	-1	-1	0	0	0
3	c	0	1	0	1	-1	0	0	0	-1	0
4	d	1	-1	1	0	1	-1	-1	0	0	0
5	e	0	1	-1	1	0	1	0	-1	0	-1
6	f	0	-1	0	-1	1	0	1	0	1	-1
7	g	0	-1	0	-1	0	1	0	1	-1	1
8	h	-1	0	0	0	-1	0	1	0	1	0
9	i	-1	0	-1	0	0	1	-1	1	0	1
10	j	0	0	0	0	-1	-1	1	0	1	0

Table 5. A mixed-tie network with zero row and column sums.

<INSERT TABLE 6 ABOUT HERE>

a	0.901
b	0.861
c	0.908
d	0.861
e	0.841
f	0.850
g	0.862
h	0.902
i	0.851
j	0.907

Table 6. PN centrality scores for the network shown in Table 5

It should be noted that PN centrality can give negative scores. The range for networks containing just positive scores is 1 to 2, the range for networks with just negative scores is 0 to 1. However the range for networks with both positive and negative ties is -1 to 2. We could, of course, make these always positive by simply adding a 1 to all the scores. (And if we wanted to do that, we might also consider first dividing the scores by 3 and then adding

1/3 so as to normalize the measure between 0 and 1.) Negative scores occur when groups of positively connected actors all have negative connections to a minority of other actors. The most extreme example would be if every actor in the network had a negative connection to one actor and positive connections to every other actor. In this case the actor with all the negative ties would have a PN centrality score of -1 and all the other actors would have the maximum score of 2. Negative scores are apparently only possible when a majority of positively connected actors all dislike a small minority of other actors.

We now examine PN centrality for directed data. Proceeding in the same way as for the undirected version namely that we extend equations 7 and 8 in the same way as we extended equation 3 to get equation 9 then we obtain equations 10 and 11, where $A = P - 2N$, as before.

$$PN_{out} = \left(I - \frac{\mathbf{1}}{4(\mathbf{n} - 1)^2} \mathbf{A} \mathbf{A}^T \right)^{-1} \left(I + \frac{\mathbf{1}}{2(\mathbf{n} - 1)} \mathbf{A} \right) \mathbf{1} \quad (10)$$

$$PN_{in} = \left(I - \frac{\mathbf{1}}{4(\mathbf{n} - 1)^2} \mathbf{A}^T \mathbf{A} \right)^{-1} \left(I + \frac{\mathbf{1}}{2(\mathbf{n} - 1)} \mathbf{A}^T \right) \mathbf{1} \quad (11)$$

In these measures, the negative ties are handled in the same way as in the h^* measures. That is, with respect to the negative ties, it is better to be disliked by people that have low h_{out}^* scores than those that have high scores, and it is better to dislike people that have low h_{in}^* scores than those that have high scores. Similarly, for the positive ties, we assume that it is better to be liked by popular people (i.e., those choosing many others), and it is better to like popular people (i.e., those chosen by many others). We also note that other combinations may be desirable, for example it may be better to be liked by people that chose very few others and/or to like people that are not as popular (exclusive relationships). There are of course a number of possible combinations that would all result in variations of the equations we have presented here. In order to keep this exposition simple we have decided to only present the model that corresponds to equations 10 and 11 and to look in more detail at the other combinations in a subsequent paper. We do note that there are 16 possible combinations if we simply look at high and low of the actors each are connected to and from.

Table 7 gives the directed PN centrality scores for the dichotomized Sampson data, we also in column 3 repeat the undirected PN scores based on the symmetrized data.

<INSERT TABLE 7 ABOUT HERE>

		PN-in	PN-out	PN
		-----	-----	-----
1	ROMUALD	0.899	1.122	1.025
2	BONAVENTURE	1.286	0.868	1.162
3	AMBROSE	1.156	0.793	1.105
4	BERTHOLD	0.739	0.625	0.944
5	PETER	0.675	0.880	0.761
6	LOUIS	0.836	0.621	0.852
7	VICTOR	0.744	0.598	0.809
8	WINFRID	1.025	0.884	0.999
9	JOHN_BOSCO	1.119	0.779	0.982
10	GREGORY	0.876	0.840	1.071
11	HUGH	0.799	0.607	0.903
12	BONIFACE	1.096	0.681	0.955
13	MARK	0.921	0.733	0.883
14	ALBERT	0.827	0.649	0.873
15	AMAND	0.648	0.682	0.665
16	BASIL	0.333	0.810	0.420
17	ELIAS	0.402	0.940	0.483
18	SIMPLICIUS	0.397	0.801	0.447

Table 7 Directed PN centrality scores for Sampson

We can now see that the low PN centrality scores for the outcasts was a consequence of the PN-in scores, namely that they were not chosen positively by many actors but were chosen negatively by many. In contrast their PN- out scores are quite high reflecting the fact that when they chose outside the group they did chose popular others.

The time complexity of exact solutions for equations 10 and 11 (as well as 7 and 8) is $O(n^3)$ as it relies on solving sets of linear equations. So the time complexity of directed h^* and PN centrality is the same as for undirected h^* . In addition, as mentioned for h^* we can construct iterative schemes based on the non-matrix formulations that can be used to compute the measures on very large networks.

7 Other values of β

We have suggested using a value for β which normalizes the centrality scores so that they range between -1 and 2. This normalization assumes that it is possible to have connections (whether positive or negative) to every other actor in the network. For larger networks this becomes unrealistic. Moreover, as network size increases, the value of $\beta = 1/(2n-2)$ tends to zero, and PN centrality becomes indistinguishable from degree. If we had a good idea of the upper bound on any node's degree we could use this number as an alternative basis for normalization. That is, if the maximum possible degree for P and for N was δ , we would set β to $(2\delta)^{-1}$. This assumes that the bound was applicable to both the positive and negative ties. If these values were different, we would need to modify the equations and have P and N separate rather combined as $A=P-2N$. Alternatively, we could use the maximum observed degree in either matrix – i.e., the maximum degree of all vertices in P or in N. We would need to check that the value of β in this case was smaller than the reciprocal of the largest eigenvalue; otherwise the matrix equations would not agree with the implicit non-matrix versions. This in turn would mean that the centrality scores are not capturing the idea that

each actor's score is proportional to the ones they are connected to. If we did chose a different β we would not be able to compare the PN scores across different networks as we would have not have normalized consistently but we would of course still be able to identify the rank ordering of the centrality scores of all the actors within the network. We do however still get a relative normalized score as the maximum value of 2 could only occur for a complete positive network and the minimum value of -1 by a network with a positive clique whose members are all connected negatively to a single node.

An alternative approach choosing a β that normalizes is search a range of β s to select one that yields the centrality scores with the largest variance. For the same reason as above it would be advisable select β to be smaller than the reciprocal of the largest eigenvalue. We do not pursue other values of β here but it is an interesting area for further research.

Conclusion

The difficulties in applying standard network concepts and methods to negative ties has been underappreciated. Some concepts are unproblematic, such as equivalence concepts. Other concepts, such as degree, require adjustments in interpretation but are otherwise applicable to both positive and negative tie data. Still other concepts, notably those that rely on notions of network flow for interpretability, are effectively unusable. These include measures of closeness and betweenness. We explore the trick of converting negative ties to positives by taking the graph complement, but find that this is rarely useful.

In view of these issues, we develop some concepts specifically designed for negative ties. The negative clique concept turns the idea of a conventional clique inside out, defining a smallest set of nodes that everyone else in the network has a negative tie with at least one member of the negative clique. We mentioned one possible application was in organizational restructuring. A second example would be in small setting such as a classroom. A group of pupils all of whom were disliked by people outside the group would all be potential candidates for bullying. Support from outside the group is unlikely as everyone outside has a negative relation to those in the group and supporting group members would be viewed badly by those outside.. An intervention that promotes mutual support within the group may help potential victims and reduce the likelihood of one individual being singled out.

We also introduce h^* , a negative centrality measure in which a node gets a high centrality score if they have few negative ties to central others, this can be viewed as an extension of negative degree centrality. We further show that this new measure has the same basic form as the measure proposed by Hubbell but using a different sign for the attenuation factor. This fact allows us to combine Hubbell type measures for positive ties with h^* measures for negative ties to obtain PN centrality. Furthrmore we develop directed versions of these measures using arguments similar (but with a different mathematical basis) as the Hubs and Authorities measure of Kleinberg. We note that PN centrality is the only published measure that can deal with directed valued mixed data.

The original Hubbell type measures are based on the idea of flow for positive ties and so it would seem odd to have similar measures for negative ties in which flow is difficult to interpret. Since an actor's h^* is based on the scores of the actors they are connected to then these scores propagate through the network. It follows that what is flowing through the negative tie network is the measure itself, rather than a measure trying to capture anything flowing. The same argument applies to the PN centrality in which the P matrix part of the

measure can be interpreted as either capturing flow or as propagating the measure through the network. The N part of the measure can only really have the latter explanation.

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Appendix

Relationships between Hubbell, Katz, Bonacich Beta and eigenvector Centrality

As already noted in the main text, Hubbell's measure \mathbf{h} is given by

$$\mathbf{h} = (I - \beta A)^{-1} \mathbf{1}$$

Katz's measure \mathbf{k} is given by

$$\begin{aligned} \mathbf{k} &= ((I - \beta A)^{-1} - I) \mathbf{1} \\ &= (I - \beta A)^{-1} \mathbf{1} - \mathbf{1} \\ &= \mathbf{h} - \mathbf{1} \end{aligned}$$

Hence, Katz's measure is Hubbell's minus one.

Bonacich's beta centrality (un-normalized) is given by

$$\mathbf{b} = (I - \beta A)^{-1} A \mathbf{1}$$

Now,

$$\begin{aligned} \mathbf{k} &= ((I - \beta A)^{-1} - I) \mathbf{1} \\ (I - \beta A) \mathbf{k} &= (I - (I - \beta A)) \mathbf{1} \\ (I - \beta A) \mathbf{k} &= \beta A \mathbf{1} \\ \mathbf{k} &= \beta (I - \beta A)^{-1} A \mathbf{1} \\ \mathbf{k} &= \beta \mathbf{b} \end{aligned}$$

Thus, Katz's measure is just β times Bonacich's.

Finally we note that Bonacich (1987) showed that as $\beta \rightarrow 1/\lambda_{\max}$ then $\mathbf{b} \rightarrow$ eigenvector centrality.