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An extension of regular colouring of graphs to digraphs, networks and hypergraphs *

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The use of regular graph colouring as an equivalent simple definition for regular equivalence is extended from graphs to digraphs and networks. In addition new concepts of regular equivalence for edges and hypergraphs are presented using the new terminology.

1. Introduction

Everett and Borgatti (1991) have suggested an alternative formulisation of Regular Equivalence (White and Reitz 1983) using similar ideas to classical graph colouring. This formulisation provides a simple representation of regular equivalence without the need to resort to complicated algebraic constructions. In this paper we shall extend the ideas in Everett and Borgatti's paper to digraphs and networks in keeping with White and Reitz's original usage of regular equivalence. In addition, using the new terminology, we examine how regular equivalence can be extended to edges and hypergraphs.

We first introduce the idea of regular colouring for graphs. (Note that Everett and Borgatti (1991) call this role-colouring.) Let $G(V, E)$ be an undirected graph (self-loops are permissible) with vertex set V and edge let E. The *neighbourhood* $N(v)$ of a vertex $v \in V$ is the set of all verticles adjacent to v. So that $N(v) = \{x:(v, x) \in E\}$.

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* This paper is based on Chapters 2 and 3 of Borgatti (1989).

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A *colouring* of a graph G is an assignment of colours to the vertices of G: if $v \in V$ then we denote the colour of v by $C(v)$.

If S is a subset of the vertices of a graph then we define the *colour set* of S written $C(S)$, (or *spectrum* of S) as the set of all colours assigned to the vertices of S.

If we examine the coloured graph in Fig. 1 we see that vertex 1 has been coloured red, vertices, 2, 4 and 5 green and the remainder black. The neighbourhood of vertex 2 is the set of vertices {1, 4, 5}, i.e. $N(2) = \{1, 4, 5\}$. The spectrum of 1 is red; i.e. $C(1) = \{\text{Red}\}$, the spectrum of {1, 6, 7} is {red, black}. We shall require the spectrum of a neighbourhood e.g. $C(N(2)) = \{\text{red, green}\}$.

The colouring of a graph induces a partition of the vertices. We assume all vertices coloured the same belong to the same class.

A colouring of a graph is a *regular colouring* if whenever two vertices are coloured the same the spectrum of their neighbourhoods is the same, i.e. $G(V, E)$ is regularly coloured if and only if for all v, $w \in V$

$$C(v) = C(w) \Rightarrow C(N(v)) = C(N(w)).$$

If we examine Fig. 1 we see that vertex 6 and vertex 7 are coloured the same (black) and that $C(N(6)) = C(N(7)) = \{\text{green, black}\}$, hence this pair of vertices satisfies the condition. However, we do not have a

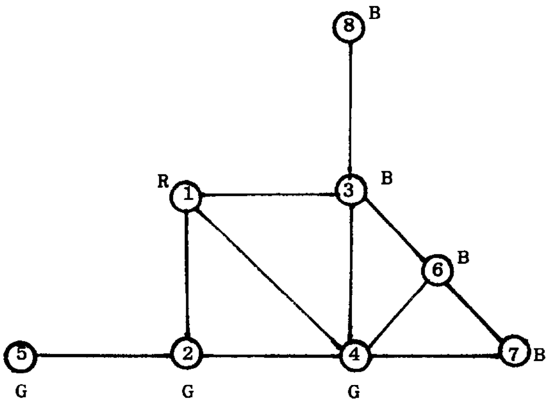

Fig. 1.

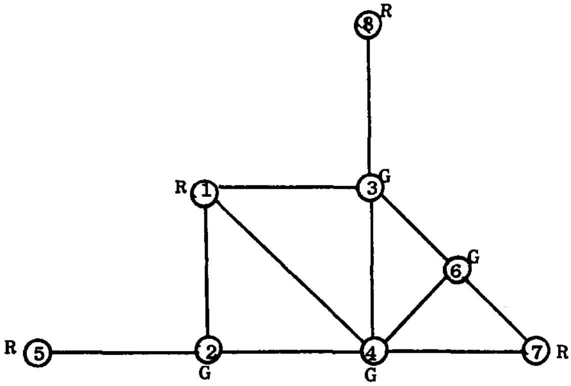

Fig. 2.

regular colouring since there are other black vertices whose neighbourhoods have a different spectrum; we see for example that $C(N(8)) = \{\text{black}\}$.

An alternative colouring, which is regular is shown in Fig. 2. In this colouring the spectrum of the neighbourhood of any red vertex is green, i.e. $C(N(1)) = C(N(5)) = C(N(7)) = C(N(8)) = \{\text{green}\}$ and the spectrum of the neighbourhood of any green vertex is red and green. i.e. $C(N(2)) = C(N(3)) = C(N(4)) = C(N(6)) = \{\text{green, red}\}$.

The *colour-image graph* $G'(C(V), E')$ of a coloured graph $G(V, E)$ has the spectrum of V as its vertices; two vertices are adjacent in G' if there exists an edge between the colours in G. Figure 3 is the colour-image graph of Fig. 1 and Fig. 4 is the colour-image graph of Fig. 2. If we examine Fig. 3 we see that the edge connecting green and

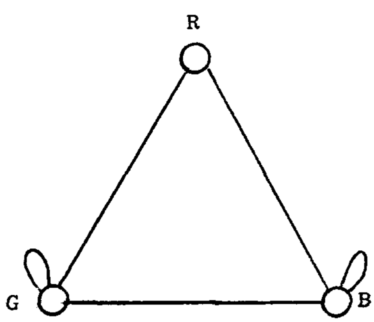

Fig. 3.

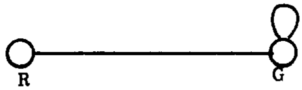

Fig. 4.

red implies the existence of a red–green edge in Fig. 1 (for example {1, 2} or {1, 4}), but note that there may be green vertices which are not connected to red vertices (vertex 5 for example). In a regular colouring this cannot occur; in Fig. 4 the fact that red and green are adjacent means that every red must be adjacent to some green and every green must be adjacent to some red. This fact is further explored in the following theorem.

Theorem 1. Let $G(V, E)$ be a coloured graph and $G'(C(V)E')$ be the colour-image graph. Then the colouring of G is a regular colouring if and only if for all $v \in V$, $C(N(v)) = N(C(v))$. [Note that for each vertex v, $C(v)$ will be a vertex in G'.]

Proof. We note that in the colour-image graph G', two colours are adjacent if there exists an edge connecting the colours in G; it follows that for all $v \in V$ $C(N(v)) \subset N(C(v))$. Suppose that the colouring is regular and there exists a $v \in V$ such that $C(N(v)) \neq N(C(v))$. Hence there is some colour A s.t. $A \in N(C(v))$ but $A \notin C(N(v))$. If $A \in N(C(v))$ then there exists $y \in V$ s.t. $C(y) = C(v)$ and $A \in C(N(y))$. Since we know that $A \notin C(N(v))$ this contradicts the regularity of the colouring.

Conversely suppose that for all $v \in V(C(N(v)) = N(C(v))$ if $C(x) = C(y)$ then $N(C(x)) = N(C(y))$ and hence $C(N(x)) = C(N(y))$ and the colouring is regular.

2. Digraphs

The extension to digraphs is reasonably straight forward, we just need to take account of directionality in the construction of the neighbourhoods. Let $G(V, E)$ be a directed graph. We defined the in-neighbourhood of a vertex v as the set of vertices from which v receives connections, and the out-neighbourhood as the set of vertices

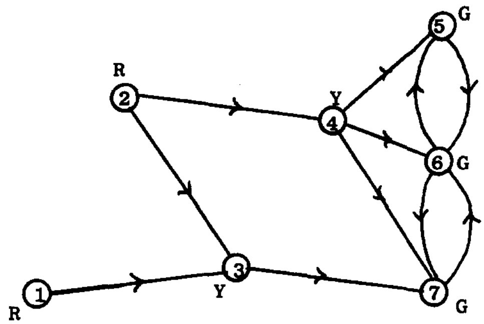

Fig. 5.

which receive connections from v; these are denoted by $N_i(v)$ and $N_o(v)$, respectively. So that

$$N_i(v) = \{x: (x, v) \in E\}$$

$$N_o(v) = \{x: (v, x) \in E\}.$$

We can now define a regular colouring of a digraph G as a colouring in which if two vertices are coloured the same then their respective in-neighbourhoods and out-neighbourhoods must have the same spectrum, i.e. $G(V, E)$ is regularly coloured if and only if for all $v, w \in V$

$$C(v) = C(w) \Rightarrow C(N_i(v)) = C(N_i(w)) \text{ and}$$

$$C(N_o(v)) = C(N_o(w)).$$

If we examine the digraph in Fig. 5 we see that $N_o(4) = \{5, 6, 7\}$, $N_i(4) = \{2\}$ and $N_i(2) = \emptyset$ etc. In fact the colouring is regular since the red vertices are connected to yellow, the yellow vertices receive from reds and connect to greens, green vertices receive from yellows and are connected to greens.

The underlying graph of a digraph is the graph that results when we remove the direction of the arcs. Figure 6 gives the underlying graph of Fig. 5.

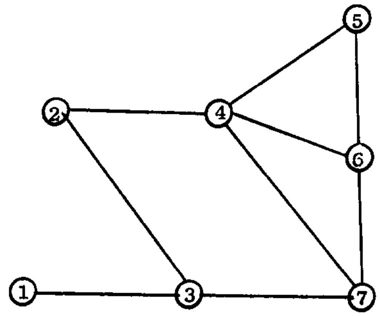

Fig. 6.

Theorem 2. Any regular colouring of a digraph is also a regular colouring of the underlying graph.

Proof. The result follows from noting that the neighbourhood of any vertex in the underlying graph is simply the union of the in-neighbourhood and out-neighbourhood of the corresponding vertex in the digraph.

The usefulness of this theorem becomes apparent when we note that it can be used in reverse. The algorithm REGE finds the maximal regular equivalence of a digraph, for a graph, without isolates, this is the trivial partition in which every vertex is equivalent. The above theorem means that we can take a graph, form a directed graph by placing directions on the edges, submit this to REGE and the result will be a regular equivalence for the original graph. The process of attributing directions to an undirected graph is call *orientation*. The technique works for any random orientation but it would obviously be advisable to use an orientation which is interpretable. Directions could, for example, be associated by orientating each edge from less central to more central vertices. This approach was suggested by Doreian (1987); although in a slightly different implementation. In practice the orientated digraph would probably not contain any regularly equivalent actors, and so clustering on the REGE measure of equivalence would be required. The above theorem merely legitimises this action on the grounds that when perfect equivalence does occur it is consistent.

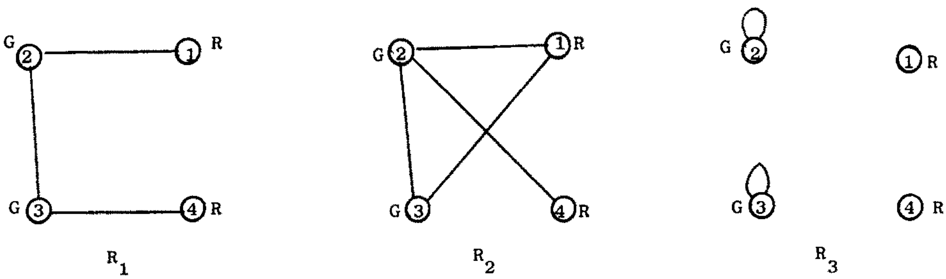

Fig. 7.

3. Networks

All the material in this section is a restatement of work first done by Doug White and Karl Reitz (White and Reitz 1983; Reitz and White 1989). A network is a collection of graphs or digraphs with a common vertex set. We shall denote a network with vertex set V and edge sets $\{R_i\}_{i \in I}$ by $G(V, \{R_i\}_{i \in I})$ or simply $G(V, \{R\})$. A colouring of the vertices V or a network $G(V, \{R_i\}_{i \in I})$ is a *regular network colouring* if and only if $G(V, R_i)$ is a regular colouring for every $i \in I$. In other words, we insist that each graph taken separately has a regular colouring. In constructing the colour-image network we form the colour-image graph for each relation R_i and then delete any isomorphic colour-image graphs. These concepts are illustrated on the three relation network shown in Fig. 7. The colouring of the vertices is regular on each relation R_1, R_2, R_3 hence the colouring is a regular network colouring. The individual images R'_1, R'_2 and R'_3 are shown in Fig. 8.

We see that R'_1, R'_2 are identical and hence the colour-image network consists of the two relations shown in Fig. 9.

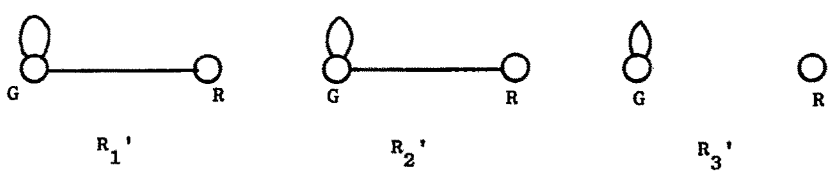

Fig. 8.

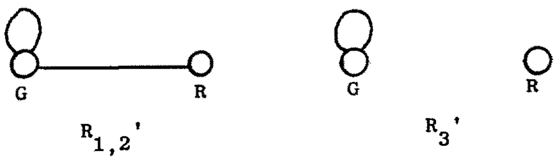

Fig. 9.

We can think of $R'_{1,2}$ as the union to the two relations R'_1 and R'_2. It follows that we could obtain exactly the same colour-image network by taking the union of R_1, R_2, in the original network. Under the given colouring the relations R_1 and R_2 are equivalent in as much as they produce the same colour image. Given a vertex colouring of a network then the colour-reduced network is the network formed by taking the union of all relations with identical colour-image graphs. We denote the colour-reduced network of a network G by $CR(G)$. The $CR(G)$ of the network shown in Fig. 7 is given in Fig. 10.

It is always true that the colour-image network of G and the colour-image network of $CR(G)$ are identical. A *weak regular colouring* of a network G is a network regular colouring of $CR(G)$. Every network regular colouring is a weak regular colouring but the converse is not true. Figure 11 gives a weak regular colouring for a network which is not a regular vertex colouring.

The colour-image network is given in Fig. 12 and we can see that R'_1 and R'_2 are identical. It follows that the colour-reduced graph

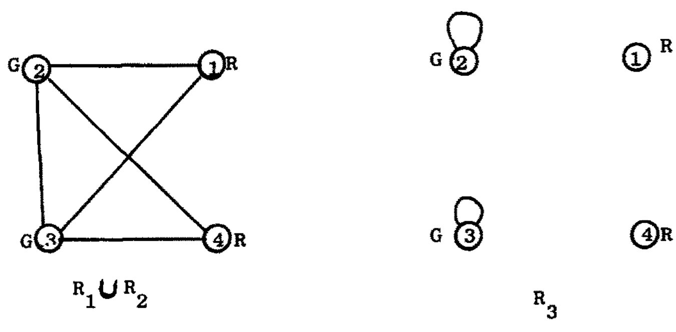

Fig. 10.

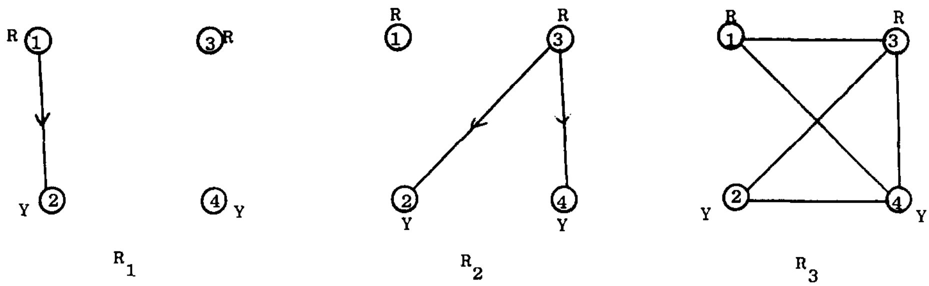

Fig. 11.

given in Fig. 13 unions together the relations R_1 and R_2. It can easily be seen that the colouring of the colour-reduced graph is a regular network colouring and hence the colouring in Fig. 11 is a weak regular colouring. In a weak regular colouring two vertices are equivalent (i.e. coloured the same) if they are connected to equivalent others on equivalent relations.

It may happen that in a network with many relations there are many different sets of identical colour images. The colour-reduced network would collapse each set to a single image. A slightly stronger regular network colouring could be formed by only collapsing certain sets. In this case the resultant network colouring would be weakly regular on the whole network but on a subset of the relations it would still form a regular network colouring.

Regular network colouring takes no account of the different bundles of relations which may exist between equivalent vertices. Take,

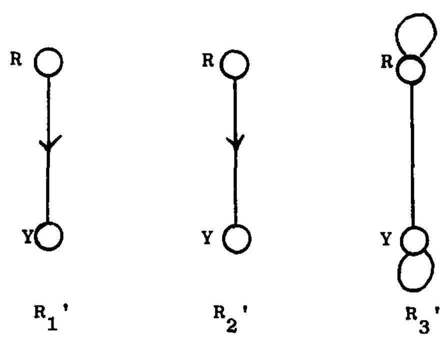

Fig. 12.

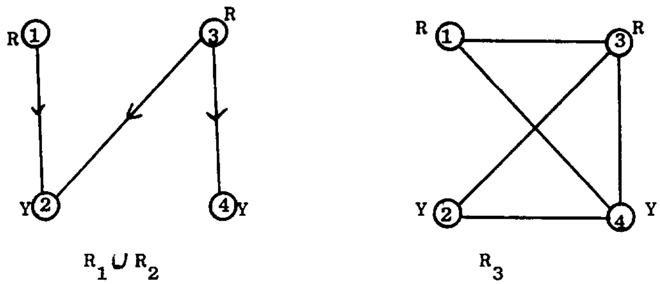

Fig. 13.

for example, the network given in Fig. 7. Consider the connections between the green vertex 2 and the red vertices 1 and 4. The 2,1 pair are connected by R_1 and R_2, and the 2,4 pair by just R_2. Now consider the connection between the other green vertex 3 and the red vertices 1 and 4. The 3,1 pair are connected by R_2 only and the 3,4 pair by R_1 only. It follows that the sets of relations between these equivalent vertices are very different. If we wish these sets or bundles of relations to be the same we need to strengthen our concept of regular network colouring. We shall use a similar trick as for the weak regular colouring and form a new network from our existing network. This time our network can contain more relations and this will make the definition more strict. The bundle of relations from v to w in a network $G(V, \{R_i\})$ is the set of relations B_{vw} which connect v to w, i.e.

$$B_{vw} = \{R_i vR_i w\}$$

Let $\{M_i\}_{i \in I}$ be the set of all non-empty bundles. We can associate with each M_i a digraph with vertex set V and edge set defined by $vM_i\text{w} = B_{vw}$. Strictly speaking each M_i is a set of relations, we have defined a new relation (which by an abuse of notation we have also called M_i) for each bundle. Two vertices are related by M_i if they are related by each relation in the bundle. Since each M_i forms a digraph on the same vertex set V these can be considered as a new network which we denote by MPX(G). If we start with a network on 3 relations R_1, R_2, R_3, then MPX can contain up to 7 relations. These correspond to vertices which are connected by R_1 only, R_2 only, R_3 only,

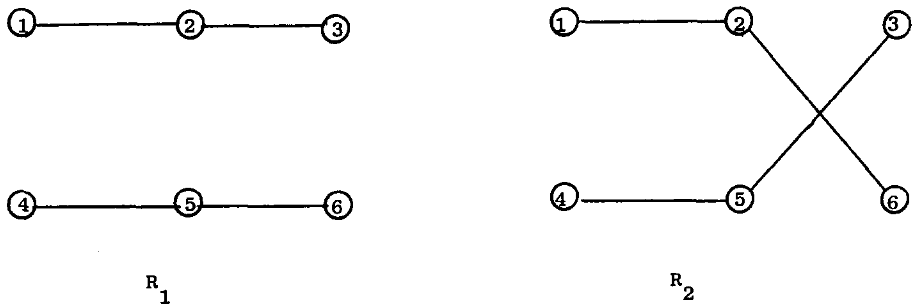

Fig. 14.

R_1 and R_2 only, R_1 and R_3 only, R_2 and R_3 only and R_1, R_2 and R_3 all together. In general if a network has n relations then MPX can have up to $2^n - 1$ relations. If two relations are identical then they will be merged in MPX, so it is possible for the MPX network to have less relations than the original network. If G is a network then a *multiplex regular colouring* of G is a colouring of G which is regular on MPX(G). Consider the network shown in Fig. 14. The MPX network with $M_1 = R_1$ relations only, $M_2 = R_2$ relations only and $M_3 =$ both R_1 and R_2 relations is shown in Fig. 15. Also in Fig. 15 is a regular network colouring for the MPX graph and hence this is *multiplex regular colouring* for the network in Fig. 14. We note the following about the multiplex regular colouring. Firstly the colouring is a network regular colouring on the original network. We shall prove this general result in Theorem 3 below. We also note that there are network regular colourings which are not multiplex regular colourings. For example the partition {1, 3, 4, 6}, {2, 5} induces a regular network colouring which is not regular on the MPX network. The network in Fig. 14 contains no isolates in any relation and hence the maximal

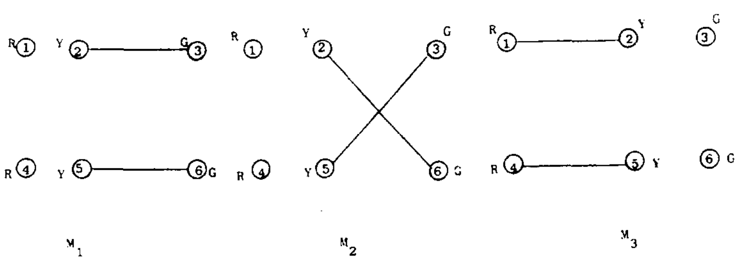

Fig. 15.

regular equivalence is the the trivial partition in which all vertices are grouped together. On the other hand the MPX graph does contain isolates, and in fact the regular colouring given is maximal on this network. Hence, the maximal multiplex regular colouring of a network maybe non-trivial even when the network does not contain isolates.

Theorem 3 (White and Reitz). Every multiplex regular colouring is a network regular colouring.

Proof. Let G be a network with a multiplex regular colouring. Consider an arbitrary relation R in G. Suppose that two vertices u, w of G are coloured the same and that x is a vertex in the R neighbourhood of u. Since uRx then there exists a relation M in MPX(G) such that uMx. It follows that $C(x)$ is a colour in the M-neighbourhood of u and since the colouring is regular on M then there exists a vertex of the same colour in the M-neighbourhood of w. That is $C(x) = C(y)$ and wMy; but if wMy then since M contains the relation R in its bundle wRy in G. It follows that there is a vertex coloured the same as x in the neighbourhood of w. Since this is true for all vertices, all colours, and all types of neighbourhood (i.e. undirected, in-neighbourhoods and out-neighbourhoods) the result follows.

4. Hypergraphs

The vast majority of techniques developed for social networks analysis is applied to data represented as graphs. As an alternative Seidman (1981) suggested that hypergraphs provide a suitable model for non-dyadic relationships. In general hypergraphs examine the situation in which a collection of subsets of the population has been identified. These subsets may be the result of some analysis on a dyadic set of data, or could be naturally occurring non-dyadic relationships. The most common non-dyadic relationships are of the actor–event type. We have a group of actors (the population) and a number of separate events; the subgroups are the groups of actors attending each event. An example is given by the data collected by Davis *et al.* (1941) which represents observed attendance at 14 social events by 18 woman. This data is given as an incidence matirix in Fig. 16. The rows correspond to the 18 women and the columns to the 14 events. A 1 in row i, col j means that woman i attended event j. We can think this data as an

		Social Events 1	2	3	4	5	6	7	8	9	10	11	12	13	14
	1	1	1	1	1	1	1	0	1	1	0	0	0	0	0
	2	1	1	1	0	1	1	1	1	0	0	0	0	0	0
	3	0	1	1	1	1	1	1	1	1	0	0	0	0	0
	4	1	0	1	1	1	1	1	1	0	0	0	0	0	0
	5	0	0	1	1	1	0	1	0	0	0	0	0	0	0
	6	0	0	1	0	1	1	0	1	0	0	0	0	0	0
	7	0	0	0	0	1	1	1	1	0	0	0	0	0	0
Women	8	0	0	0	0	0	1	0	1	1	0	0	0	0	0
	9	0	0	0	0	1	0	1	1	1	0	0	0	0	0
	10	0	0	0	0	0	0	1	1	1	0	0	1	0	0
	11	0	0	0	0	0	0	0	1	1	1	0	1	0	0
	12	0	0	0	0	0	0	0	1	1	1	0	1	1	1
	13	0	0	0	0	0	0	1	1	1	1	0	1	1	1
	14	0	0	0	0	0	1	1	0	1	1	1	1	1	1
	15	0	0	0	0	0	0	1	1	0	1	1	1	1	1
	16	0	0	0	0	0	0	0	1	1	1	0	1	0	0
	17	0	0	0	0	0	0	0	0	1	0	1	0	0	0
	18	0	0	0	0	0	0	0	0	1	0	1	0	0	0

Fig. 16.

hypergraph with a population of 18 women together with 14 subsets which consist of the women attending each event. Alternatively, we can view the data as a population of 14 events with 18 subsets which consist of those events attended by each women. The two hypergraphs formed in this way are duals of each other and are alternate ways of viewing the same data. We should expect any generalisation of regular colouring to yield consistent groupings regardless of which of the two possible duals are taken. We now present some formal definitions. Let V be a set and A a collection of non-empty subsets of V. Then $H(V, A)$ is a *hypergraph* provided every member of V is an element of one of the sets in A. We call the elements of V the vertices and the members of A the edges of the hypergraph.

Since we require our regular colourings to be consistent with our previous definition of regular colouring we must first examine the situation in which the hypergraph is a graph. In this case our non-empty subsets of V are precisely the edges of a graph G. We require a consistent colouring for the dual hypergraph; unfortunately the dual hypergraph of a graph is not necessarily a graph (in fact it nearly always is a hypergraph) and so we examine, instead, the induced colouring on the edges of G (since these will form the vertices of the

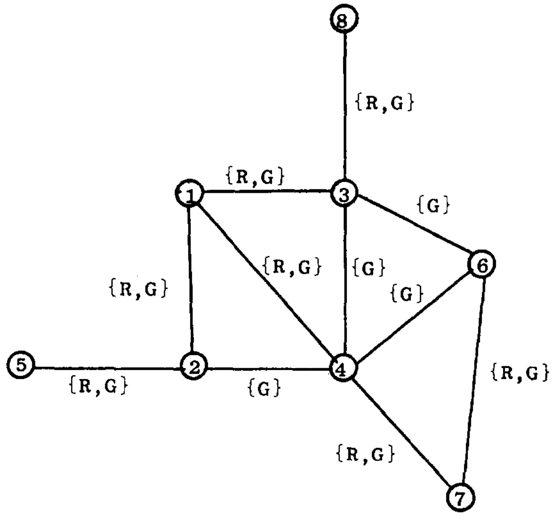

Fig. 17.

dual). Let $G(V, E)$ be a graph or hypergraph then an *edge colouring* of G is an assignment of colours to the edges of G. If $x \in E$ is an edge of an edge-coloured graph then we denote the colour of x by $C^e(x)$. The *neighbourhood* $N(x)$ of an edge (as opposed to a vertex) is the set of vertices it contains. Strictly speaking this notation is not necessary since an edge is already defined as a set of vertices. However, we shall need to differentiate between the set as an entity in its own right and the vertices which make up the set.

Given a vertex colouring of a graph or hypergraph then the *induced edge colouring* is the colouring in which each edge is coloured by the set of colours in its neighbourhood. That is, for each edge x, $C^e(x) = C(N(x))$. The induced edge colouring for the graph in Fig. 2 is given in Fig. 17.

The *edge neighbourhood* $N^e(v)$ of a vertex v is the set of edges which are incident to v. In the following theorem we characterise regular colouring on graphs in terms of the induced edge colouring.

Theorem 4. Let G be a graph with vertex colouring C. If C^e is the induced edge colouring of C then C is regular if and only if for all v, $w \in V$

$$C(v) = C(w) \Rightarrow C^e(N^e(v)) = C^e(N^e(w)).$$

Proof. Suppose C is a regular colouring and $C(v) = C(w)$. Let $a \in N^e(v)$, so that $a = \{v, x\}$ where $x \in N(v)$. Now $C^e(a) = \{C(v), C(x)\}$ and since $C(x) \in C(N(v))$ and C is regular there exists $y \in N(w)$ with $C(y) = C(x)$. It follows that $b = \{w, y\} \in N^e(w)$ with $C^e(b) = \{C(w), C(y)\} = \{C(v), C(x)\} = C^e(a)$ and hence $C^e(N^e(v)) \subset C^e(N^e(w))$. Similarly $C^e(N^e(w)) \subset C^e(N^e(v))$.

Conversely suppose that $C(v) = C(w)$ $v, w \in V$ hence $C^e(N^e(v)) = C^e(N^e(w))$. If $x \in N(v)$ then $\{x, v\} \in N^e(v)$ so that $\{C(x), C(v)\} \in C^e(N^e(v))$; it follows that there exists a $y \in N(w)$ such that $\{C(x), C(v)\} = \{C(y), C(w)\}$. If $C(v) = C(w)$ then $C(x) = C(y)$ and $C(N(v)) \subset C(N(w))$. Similarly $C(N(w)) \subset C(N(v))$ and the results follows.

We can obviously consider the dual situation of being given an edge colouring and examining the induced vertex colouring. Let G be a graph or hypergraph with an edge colouring C^e. Then the induced vertex colouring is the colouring in which each vertex is coloured by the set of colours in its edge neighbourhood. That is, for each vertex v, $C(v) = C^e(N^e(v))$. If we take the induced vertex colouring of the graph in Fig. 17 we will, of course, obtain the same type of colouring as in Fig. 2. The red vertices will be coloured $\{R, G\}$ and the green vertices will be coloured $\{\{R, G\}, \{G\}\}$.

Note that the dual of Theorem 4 is false. That is, it is possible to have an edge colouring C^e such that for all edges x, y $C^e(x) = C^e(y)$ implies that $C(N(x)) = C(N(y))$ but the induced vertex colouring is not regular. A counter-example is given in Fig. 18; it is a simple matter to verify that the given colouring satisfies the condition above and that the induced vertex colouring corresponds to the non-regular partition $\{1, 2, 4, 5\}$, $\{3\}$.

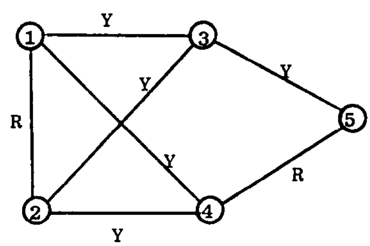

Fig. 18.

Theorem 4 is the basis for our extension of regular colourings to hypergraphs. Let $H(V, A)$ be a hypergraph with vertex colouring C and edge colouring C^e. Then the colouring is *regular* if for all vertices $v, w \in V$ and all edges $x, y \in A$.

$$C(u) = C(v) \Rightarrow C^e(N^e(u)) = C^e(N^e(v)) \quad \text{(i)}$$

$$C^e(x) = C^e(y) \Rightarrow C(N(x)) = C(N(y)). \quad \text{(ii)}$$

The first condition is precisely the one contained in Theorem 4 and the second condition ensures that the dual hypergraph is also regularly coloured. Note that the second condition holds trivially for any induced edge colouring so that any regular colouring of the vertices of a graph will be a regular colouring when the graph is considered to be a hypergraph.

The definition ensures that two vertices are coloured the same if they are adjacent to equivalent edges, and two edges are coloured the same if they connect equivalent points.

As an example consider the Davis data of Fig. 16. We shall consider a woman to be socially active if she attended more than three events. (Note that the women corresponding to rows 8, 17 and 18 do not meet this condition.) The incidence matrix of the hypergraph corresponding to the socially active women is given in Fig. 19.

		Social Events													
		1	2	3	4	5	6	7	8	9	10	11	12	13	14
Women	1	1	1	1	1	1	1	0	1	1	0	0	0	0	0
	2	1	1	1	0	1	1	1	1	0	0	0	0	0	0
	3	0	1	1	1	1	1	1	1	1	0	0	0	0	0
	4	1	0	1	1	1	1	1	1	0	0	0	0	0	0
	5	0	0	1	1	1	0	1	0	0	0	0	0	0	0
	6	0	0	1	0	1	1	0	1	0	0	0	0	0	0
	7	0	0	0	0	1	1	1	1	0	0	0	0	0	0
	9	0	0	0	0	1	0	1	1	1	0	0	0	0	0
	10	0	0	0	0	0	0	1	1	1	0	0	1	0	0
	11	0	0	0	0	0	0	0	1	1	1	0	1	0	0
	12	0	0	0	0	0	0	0	1	1	1	0	1	1	1
	13	0	0	0	0	0	0	1	1	1	1	0	1	1	1
	14	0	0	0	0	0	1	1	0	1	1	1	1	1	1
	15	0	0	0	0	0	0	1	1	0	1	1	1	1	1
	16	0	0	0	0	0	0	0	1	1	1	0	1	0	0

Fig. 19.

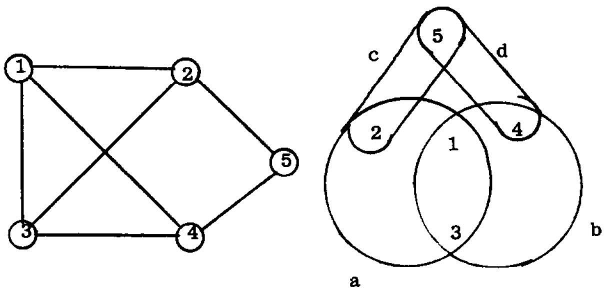

Fig. 20.

The incidence matrix has been partitioned to correspond to a hypergraph regular colouring in which the vertices (the women) are divided into two groups, and the edges into three groups. Let the women numbered 1–7 and 9 be group 1 and those numbered 10–16 be group 2. Call events 1–5 type 1, events 6–9 type 2 and events 10–14 type 3. We can see that group 1 women go to type 1 and type 2 events and group 2 women go to type 2 and type 3 events. Alternatively, or should we say dually, type 1 events are only attended by group 1 women, type 2 events are attended by both groups and type 3 are only attented by group 2 women. Note that every group 1 woman attends some (but not all) type 1 and some type 2 events. Alternatively every type 1 event is attended by at least one group 1 woman.

Hypergraphs can also be used to represent the clique structure of a graph. Figure 20 gives a simple graph together with a hypergraph representation of its cliques.

The cliques are $a = (1, 2, 3)$, $b = (1, 7, 4)$, $c = (2, 5)$ and $d = (4, 5)$ and these form the edges of the hypergraph. A regular colouring of the edges and vertices is given by the following pair of partitions.

Vertices {1, 3}, {2, 4}, {5}

Edges {a, b}, {c, d}

We again see that two vertices are coloured the same if they are members of equivalent cliques, and two cliques are coloured the same if they include equivalent members. This technique may well provide a new and useful way of analysing the clique structure of a graph.

References

Borgatti, S.P.
1989 *Regular Equivalence in Graphs, Hypergraphs, and Matrices*. Ph.D. Dissertation, University of California, Irvine.

Davis, A., B. Gardner and M. Gardner
1941 *Deep South: A Social Anthropological Study of Caste and Class*. Chicago: University of Chicago Press.

Doreian, P.
1987 "Measuring regular equivalence in symmetric structures". *Social Networks 9*: 89–107.

Everett, M.G. and S.P. Borgatti
1991 "Role colouring a graph". *Mathematical Social Science 21*: 183–188.

Reitz, K.P. and D.R. White
1989 "Rethinking the role concept: Homomorphism on social networks", in: L.C. Freeman, D.R. White and A.K. Romney (editors), *Research Methods in Social Network Analysis*. Fairfax, VA: George Mason University Press.

Seidman, S.B.
1981 "Structures induced by collections of subsets: A hypergraph approach". *Mathematical Social Sciences 1*: 381–396.

White, D.R. and K.P. Reitz
1983 Graph and semigroup homomorphism on networks of relations. *Social Networks 5*: 193–235.